1. §3.1 – 2. (Do this by the argument method, and then solve for the zeros explicitly; f is a quadratic in z^2.)

Let $f(z) = z^4 - 3z^2 + 3$. We look at the argument of $f(z)$ as z walks around a quarter-circle of radius R in the first quadrant. The first leg is from 0 to R. We have $f(x) = x^4 - 3x^2 + 3$, which reaches its minimum when $4x^3 - 6x = 0 \implies x = \sqrt{3/2} \implies f(x) = 3/4$. Thus, $f(x)$ first runs backwards from 3 to 3/4 and then streaks out rapidly to $R^4 - 3R^2 + 3$. (We are assuming here that R is large.) Now, if $z = Re^{it}$, then

$$f(z) = R^4e^{4it}\left(1 - \frac{3}{R^2e^{2it}} + \frac{3}{R^4e^{4it}}\right) = R^4e^{4it}\left(1 + \epsilon(R,t)\right)$$

where $\epsilon(R,t)$ is small. This means that, as t increases from 0 to $\pi/2$, the argument of $f(Re^{it})$ increases roughly four-fold, from 0 to 2π. (This turns out to be exact.) We now examine exactly the endpoint as we go from iR down to 0: $f(iy) = y^4 + 3y^2 + 3$ is real and positive, so, since $F(iR)$ is real, we just retreat along the u-axis to 4. The image goes around the origin once, so the argument principle says that f has exactly one root in the first quadrant. In fact,

$$z^4 - 3z^2 + 3 \implies z^2 = \frac{3 \pm i\sqrt{3}}{2} = \sqrt{3}e^{\pm i\pi/6} \implies z = \pm 3^{1/4}e^{\pm i\pi/12}.$$

2. §3.2 – 2. (Look at $|f|$ explicitly on the two line segments and the quarter circle.) Let $f(z) = ze^z$. If $z = x + iy$, then $|f(z)| = |ze^z| = |z||e^z| = |z|e^x$. It is clear, without using my hint, that $|f|$ is bounded above on the quarter circle $\{x \geq 0, y \geq 0, x^2 + y^2 \leq 4\}$ by the product of the maxima of $|z|$ and e^x separately. Since $|z| \leq 2$ and $x \leq 2$, we must have $|f(z)| \leq 2e^2$. Since these constituents share a maximum at the point $z = 2$, this is clearly the maximum of $|z|$ on this region.

As penance, I'll compute $|f(z)|$ on the boundary. As z goes from 0 to 2 on the real axis, $|f|$ clearly increases from $0e^0 = 0$ to $2e^2$. Along the quarter-circle $z = 2\cos t + i\,2\sin t$, we have $|f(z)| = 2e^{2\cos t}$, which decreases from $2e^2$ to $2e^0 = 2$ as t increases from 0 to $\pi/2$. Finally, on the imaginary axis, with $z = iy$, $|f(z)| = |z|$ decreases from 2 to 0. The point of all this is that the maximum of $|f|$ on the region occurs on the boundary (at $|z| = 2$).

3. §3.3 – 4abd. There are many different ways to do such problems, and I'll do each part differently.

First, 4a. We have $T(1)=-1$, $T(i)=i$ and $T(-1)=1$. Writing $T(z)=\frac{az+b}{cz+d}$, these imply that $\frac{a+b}{c+d}=-1$, $\frac{ai+b}{ci+d}=-i$ and $\frac{-a+b}{-c+d}=1$. The first and third imply that $a+b=-c-d, -a+b=-c+d$, and so $b=-c$ and $a=-d$. The second then implies that

$$ai+b=-i(ci+d)=c-id=-b+ai \implies b=0, c=0, d=-a,$$

so that $T(z)=-\frac{1}{z}$. (If you can guess this, more power to you.)

For 4b, we have $T(1)=0$, $T(4)=1-i$ and $T(\infty)=1+i$. Taking a different tack, we can absorb the first and third pieces of information, and use the second one to tell us that $T(z)=(1+i)\left(\frac{z-1}{z+d}\right) \implies 1-i=\frac{3(1+i)}{4+d} \implies d=-4+3i$. That is, $T(z)=(1+i)\left(\frac{z-1}{z-4+3i}\right)$.

Finally, in 4d, if $T(\infty)=1$, $T(-1)=0$ and $T(i)=1-i$, then the cross-ratio method implies that

$$\frac{z-z_1}{z-z_2}\cdot\frac{z_3-z_2}{z_3-z_1}=\frac{T(z)-w_1}{T(z)-w_2}\cdot\frac{w_3-w_2}{w_3-w_1} \implies \frac{z-\infty}{z+1}\cdot\frac{i+1}{i-\infty}=\frac{T(z)-1}{T(z)}\cdot\frac{1-i}{1-i-1}$$
$$\implies \frac{1+i}{z+1}=(1+i)\left(\frac{T(z)-1}{T(z)}\right) \implies T(z)=\frac{z+1}{z}.$$

4. §3.3 – 5bc. I think the fastest way of doing these problems is to examine the images and make a reasonable observation, rather than a systematic approach. My apologies for having two problems which each can be done in basically the same way. In 5b, we want to map the circle $|z|=1$ to the circle $|w-1|=1$, and this is basically just a translation by one unit to the left, so $w=z+1$ will clearly work. In 5c, the real axis in mapped onto the line $Re\ w=\frac{1}{2}$. Here, we need to rotate the line by 90 degrees around the origin, and then translate one-half unit to the right. This can be done by taking $w=iz+\frac{1}{2}$. And it is plain that, if z is real, then w has real part $\frac{1}{2}$.

5. ($\mathcal{E}$) Find the first three terms of the Taylor series of the Principal Value of $f(z)=z^i$ at $z=1+i$. Unevaluated expressions such as $(1+i)^i$ should not appear in your final answer. Well, as you might have done without much thought,

$$f(z)=z^i:=e^{i\log z} \implies f'(z)=e^{i\log z}(i\log z)'=e^{i\log z}\cdot\frac{i}{z}=ie^{(i-1)\log z}\left(=iz^{i-1}\right),$$

and, similarly, $(z^i)''=i(i-1)z^{i-2}=\frac{i(i-1)}{z^2}\cdot z^i$. Accordingly, the first three terms of the Taylor series (ie the power series) for $f(z)$ at $z_0=1+i$ is

$$f(z)=f(z_0)+f'(z_0)(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2.$$
$$=(1+i)^i\left(1+\frac{i}{1+i}\cdot(z-(1+i))+\frac{i(i-1)}{2(1+i)^2}\cdot(z-(1+i))^2\right)$$

Since $(1+i)^i=e^{i\ log(1+i)}=e^{i(\ln 2+i\pi/4)}=e^{-\pi/4+i\ln 2}$, one version of the correct answer is

$$e^{-\pi/4+i\ln 2}\left(1+\frac{1+i}{2}\cdot(z-(1+i))+\frac{-1+i}{4}\cdot(z-(1+i))^2\right).$$

6. and 7. ($\mathcal{E}$) Find the bilinear [i.e., linear fractional] transformation T so that $T(-1) = 0$, $T(1) = \infty$ and $T(\infty) = i$. Determine the images of the unit disk $x^2 + y^2 \leq 1$ and the upper half plane under T. Also, determine the set S with the property that T maps S onto the unit disk $u^2 + v^2 \leq 1$.

Since $T(-1) = 0$ and $T(1) = \infty$, we have $T(z) = \lambda\left(\frac{z+1}{z-1}\right)$, and $T(\infty) = i$ means that $\lambda = i$; that is,

$$T(z) = \frac{i(z+1)}{z-1}.$$

It follows from the three points given that T maps the line or circle containing $-1, 1, \infty$ (that is, the real axis) to the line or circle containing $0, \infty, i$, (that is, the imaginary axis). A calculation shows that $T(i) = \frac{i(1+i)}{i-1} = \frac{-1+i}{i-1} = 1$. Thus, T maps the upper half plane to the right half plane. Since the unit circle is determined by $-1, i, 1$, its image is the line or circle determined by $T(-1), T(i), T(1)$ (or $0, 1, \infty$); namely, the real axis. Since 0 is inside the unit circle and $T(0) = \frac{i}{-1} = -i$, it follows that the image of the interior of the unit disk is the lower half plane. To find the preimage of the unit circle, notice that $T(\infty) = i$, $T(i) = 1$ and $T(0) = -i$. Thus, the line or circle determined by $\infty, i, 0$ (that is, the imaginary axis) is mapped to the unit circle. Since -1 goes to a point inside the unit circle, we see that S is the left half plane.

8. Using a calculator or computer if necessary, and Rouché's Theorem, compute the number of zeros of the function $f(z) = e^z + z^3$ on the sets $|z| \leq .5$ and $|z| \leq 3$.

Recall that $|e^z| = e^x$, where $z = x + iy$ On $|z| = \frac{1}{2}$, $|z^3| = \frac{1}{8}$, and $e^{.5} \geq |e^z| \geq e^{-.5} \approx .61 > .125$. Thus $|e^z| > |z^3|$ for $z = \frac{1}{2}$, and Rouché's Theorem implies that e^z and $e^z + z^3 = f(z)$ have the same number of zeros inside $|z| \leq .5$; namely, none.

On $|z| = 3$, we have $3^3 = |z|^3 > e^3 \geq |e^z|$, hence Rouché's Theorem implies that z^3 and $e^z + z^3 = f(z)$ have the same number of zeros inside $|z| \leq 3$; that is, 3.

Warning: if we wanted to look at $|z| = 10$, say, then Rouché's Theorem does not apply: $|z|^3 = 1000$, but $|e^z|$ ranges from $e^{10} \approx 2 \times 10^4$ to $e^{-10} \approx 5 \times 10^{-5}$. No information is given, even though obviously f must have at least three zeros within $|z| \leq 10$. Mathematica tells us that f has three zeros, at $z \approx -.7728$ and at $z \approx .1846 \pm 1.047i$. Mathematica also tells us that $g(z) = e^z - z^3$ has four zeros, at $z \approx 1.857$, $z \approx 4.5364$ and at $z \approx -.5544 \pm .6194i$. The Rouché Theorem information, as used above, cannot distinguish between $e^z + z^3$ and $e^z - z^3$, and so is at best a blunt tool in counting zeros.

9. and 10. ($\mathcal{E}$) Let $T(z) = \dfrac{i}{z+1}$.

a. Suppose $c > 0$ is a fixed real number. Determine the curve in the w-plane that is the image of the line $x = c$ under the mapping $w = T(z)$. Be specific about the equation satisfied by u and v. If the image is a circle, specify its center and radius.
a. There are lots of ways to do this problem. We can write out the components of

$$w = u + iv = \frac{i}{(c+iy)+1} = \left(\frac{y}{(1+c)^2+y^2}\right) + i\left(\frac{1+c}{(1+c)^2+y^2}\right)$$
$$\implies u^2+v^2 = \frac{1}{(1+c)^2+y^2} = \frac{v}{1+c},$$

or we can work with as few computations as possible. Notice that T is a composition of three linear fractional transformations:

$$z \mapsto z+1 \mapsto \frac{1}{z+1} \mapsto \frac{i}{z+1}.$$

The line $x = c$ gets mapped in the first to the line $x = c+1$, since it is shifted to the right by one unit. Then, when we take the inversion, there are two cases. If $c = -1$, the line $x = 0$ (the imaginary axis) is mapped to the imaginary axis. Otherwise, the line $x = c+1$ is mapped to the circle with center $\left(\frac{1}{2(c+1)}, 0\right)$ and radius $\frac{1}{2(c+1)}$. The final map is a rotation by 90 degrees, so, if $c = -1$, the image is the line $y = 0$ and if $c \neq -1$, then the image is the circle with center $\left(0, \frac{1}{2(c+1)}\right)$ and radius $\frac{1}{2(c+1)}$. Put in terms of the components, this is the familiar equation

$$u^2 + \left(v - \frac{1}{2(c+1)}\right)^2 = \left(\frac{1}{2(c+1)}\right)^2 \implies (1+c)(u^2+v^2) = v.$$

Which is"correct"? Wrong question: you are better off knowing both methods.
b. Determine and sketch the image of the half-strip $0 \leq x \leq 1$, $y \geq 0$. Indicate on your picture the following five points: $T(0)$, $T(1)$, $T(i)$, $T(1+i)$ and $T(\infty)$.

Omitting routine computations, we have

$$T(0) = i, \qquad T(1) = \frac{i}{2}, \qquad T(i) = \frac{1+i}{2}, \qquad T(1+i) = \frac{1+2i}{5}, \qquad T(\infty) = 0.$$

We know from (a) that the image of the line $x = 0$ is the circle centered at $(0, \frac{1}{2})$ with radius $\frac{1}{2}$ and the image of the line $x = 1$ is the circle centered at $(0, \frac{1}{4})$ with radius $\frac{1}{4}$. The image of the real axis is the imaginary axis. The resulting comma-like figure is sketched below.