1. §2.1 – 14 (The easiest proof is by induction on n.) By the remark, I meant that you can give an indication of the truth by writing formulas with ..., but we really need induction to establish an n-fold product rule. In any event, if $n = 1$, and $p(z) = A(z - z_1)$, then $p'(z) = A$, so $\frac{p(z)}{p'(z)} = \frac{1}{z-z_1}$. Supposing the formula is correct for products of $n - 1$ linear factors, we have $p(z) = \tilde{p}(z)(z - z_n)$, where $\tilde{p}(z) = A \prod_{k=1}^{n-1}(z - z_k)$. (Note: index changed from j to k in deference to the engineers in class, who write i as j, because they've already decided that i is current, and can't be a complex number, but I digress.) In any cases, by the ordinary product rule and the induction hypothesis, we have for $z \neq z_j$,

$$p'(z) = (\tilde{p})'(z)(z - z_n) + \tilde{p}(z) \cdot 1 \implies \frac{p'(z)}{p(z)} = \frac{(\tilde{p})'(z)(z - z_n)}{\tilde{p}(z)(z - z_n)} + \frac{\tilde{p}(z)}{\tilde{p}(z)(z - z_n)}$$
$$= \sum_{k=1}^{n-1} \frac{1}{z - z_k} + \frac{1}{z - z_n} = \sum_{k=1}^{n} \frac{1}{z - z_k}.$$

2. §2.1 – 20 c. Let $u = 2x^2 + 2x + 1 - 2y^2$. then $u_x = 4x + 2$ and $u_y = -4y$. By the Cauchy-Riemann equation, we have $v_x = -u_y = 4y$, so that $v(x, y) = 4xy + h(y)$ for some function h that depends only on y. But then, $v_y = 4x + h'(y) = u_x = 4x + 2$, hence $h'(y) = 2$, so $h(y) = 2y + c$ for some constant c and $v(x, y) = 4xy + 2y + c$. This constant is real because u and v are real functions. Putting this together, and going beyond what the question asked, we have

$$f(z) = f(x + iy) = u(x, y) + iv(x, y) = 2x^2 + 2x + 1 - 2y^2 + i(4xy + 2y + c)$$
$$= 2(x + iy)^2 + 2(x + iy) + 1 + ic = 2z^2 + 2z + 1 + ic.$$

3. §2.2 – 2. A desperate appeal to the ratio test. We have $a_k = \frac{(k!)^2}{(2k)!}$; as $k \to \infty$,

$$\frac{a_{k+1}}{a_k} = \frac{\frac{((k+1)!)^2}{(2k+2)!}}{\frac{(k!)^2}{(2k)!}} = \frac{((k+1)!)^2}{(k!)^2} \cdot \frac{(2k)!}{(2k+2)!} = \frac{(k+1)^2}{(2k+1)(2k+2)} = \frac{k+1}{4k+2} \to \frac{1}{4}.$$

It follows that the radius of convergence of the series is 4; that is, the series $\sum_{k=0}^{\infty} a_k(z-2)^k$ converges for $|z - 2| < 4$ and diverges for $|z - 2| > 4$.

4. §2.2 – 8. Just playing around with the formulas you know and love and can find in the book:

$$\cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} \implies z^2 \cos z = z^2 \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+2}}{(2k)!}.$$

If you are a fan of the dot-dot-dot, this is

$$z^2 \cos z = z^2 - \frac{z^4}{2} + \frac{z^6}{24} - \frac{z^8}{720} + - \ldots.$$

§2.2 – 10. Even easier, in a sense. This comes from multiplying the series (for $|z| < 1$):

$$\frac{1+z}{1-z} = (1+z)\sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty} z^k + z\sum_{k=0}^{\infty} z^k = 1 + 2\sum_{k=1}^{\infty} z^k = 1 + 2z + 2z^2 + \ldots.$$

Another (and equally valid) way to get this formula is to observe that $\frac{1+z}{1-z} = \frac{2-(1-z)}{1-z} = \frac{2}{1-z} - 1 = 2\sum_{k=0}^{\infty} z^k - 1$.

5. Determine the image of the region

$$A = \{(x, y) \ : \ x < 1 \quad \text{and} \quad y < 1\}$$

under the map $w = f(z) = 1/z$. The fastest way to answer this question is to observe that the mapping is one-to-one and A is the complement of $B \cup C$, where $B = \{(x, y) \ : \ x \geq 1\}$ and $C = \{(x, y) \ : \ y \geq 1\}$. A picture will show that C is B under the mapping $z \mapsto iz$. Now we know from Homework 2,#7,9 that the line $x = x_0$ maps to the circle with center $\frac{1}{2x_0}$ and radius $\frac{1}{2x_0}$ (with the understanding that "∞" lies on the line and maps to 0; each such circle that passes through the origin). Taking the union of all such circles for $x_0 \geq 1$ gives us the solid disk with center $\frac{1}{2}$ and radius $\frac{1}{2}$. This disk is the image of B. We could repeat the reasoning of Homework 2 to find the image of C, or we can be sneaky. Since $C = iB$, in the sense that $z \in C \iff z = iz', z' \in B$, we know that the image of C under $w = 1/z$ will be $1/i$ times the image of B. Thus, the image of C will be the circle with center $-\frac{i}{2}$ and radius $\frac{1}{2}$. (The intersection of the two circles is the image of $B \cap C$, or $\{x > 1, y > 1\}$.) What we want is the **complement** of the image of $B \cup C$ (since f is one-to-one and onto), and so is the exterior of the union of the two circles.

There are many ways to do this problem, and I will read all your solutions carefully. It is not really sufficient to look only at the image of the boundary.

6. ($\mathcal{E}$) Show that the function $u(x,y) = y^3 - 3x^2y + 3x$ is harmonic, and calculate any harmonic conjugate $v(x,y)$ by any correct method. Express $f(z) = f(x+iy) := u(x,y) + i \cdot v(x,y)$ as a function of z alone.

By "any", I mean that we can set constants equal to zero if we like. I like. We're given $v_x = -u_y = -3y^2 + 3x^2 \implies v = -3xy^2 + x^3 + h(y)$. Thus, $v_y = u_x = -6xy + 3 = -6xy + h'(y)$. Therefore, $h'(y) = 3$, and since we can pick any solution, we'll take $h(y) = 3y$, so $v(x,y) = -3xy^2 + x^3 + 3y$. Putting it together,

$$f(z) = f(x+iy) = u(x,y) + iv(x,y) = y^3 - 3x^2y + 3x + i(-3xy^2 + x^3 + 3y).$$

We are suspicious that $z^3 = (x+iy)^3 = x^3 + 3ix^2y + 3i^2xy^2 + i^3y^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$ ought to show up, and in fact, it's easy to see that $f(z) = iz^3 + 3z$.

7a. Find, carefully, all complex numbers z with the property that $e^z = 4 + 4i$. This is just a way of asking to find all values of $\log(4+4i)$. Since $4 + 4i = (4\sqrt{2})e^{\pi i/4}$, we have $z = \ln(4\sqrt{2}) + i(\frac{\pi}{4} + 2\pi k)$, where $k \in \mathbf{Z}$.

7b. Find, carefully, all complex numbers z with the property that $\sin z = 10$. The point of "carefully" here is that it's not enough to just use the arcsine formula, but you need all values. We have:

$$10 = \sin z = \frac{e^{iz} - e^{-iz}}{2i} \implies \left(e^{iz}\right)^2 - (20i)\left(e^{iz}\right) - 1 = 0$$
$$\implies e^{iz} = \frac{20i \pm \sqrt{(-10i)^2 + 4}}{2} = (-10 \pm \sqrt{99})i.$$

Now, in either case of sign, $-10 \pm \sqrt{99} < 0$, so

$$|(-10 \pm \sqrt{99})i| = 10 \mp \sqrt{99}; \qquad Arg((-10 \pm \sqrt{99})i) = -\tfrac{\pi}{2}i.$$

Taking the logarithm, we have

$$iz = \ln(10 \mp \sqrt{99}) - \frac{\pi i}{2} + 2\pi ki, \qquad k \in \mathbf{Z}.$$

Upon dividing by i (or multiplying by $-i$), we find that

$$z = -\frac{\pi}{2} + 2\pi k - \ln(10 \mp \sqrt{99})i, \qquad k \in \mathbf{Z}.$$

It is reasonable, but not mandatory, to observe that $(10 + \sqrt{99})(10 - \sqrt{99}) = 1$, hence the last term above could be written as $\pm \ln(10 + \sqrt{99})i$.

8. ($\mathcal{E}$) Determine all possible value (or values) for $f(z) = Log\ ((1-i)z) - Log\ (z)$, as z ranges over the complex numbers minus the non-positive reals, and $Log\ z$ denotes the Principal Value of the logarithm. For each value w_0 that you say f takes, find a specific z_0 so that $f(z_0) = w_0$. This is, of course, very similar to Homework 3, Number 10. Note first that $1 - i = \sqrt{2}e^{-i\pi/4}$, so if $z = re^{it}, -\pi \le t < \pi$, then $(1-i)z = r\sqrt{2}e^{i(t-\pi/4)}$.

I've chosen this range for t because it makes it easy to write $Log\ (z) : \ln r + it$. What about $Log\ ((1-i)z)$? It's equal to $\ln(\sqrt{2}r) + i(t - \pi/4) + i2\pi k$, where k is chosen so that $(t-\pi/4) + i2\pi k \in [-\pi, \pi)$. Given the range of t, we see that $k = 0$ if $t \in [-3\pi/4, \pi)$ and $k = 1$ if $t \in [-\pi, -3\pi/4)$. Thus the two values for $f(z)$ are $\ln 2 - i\pi/4$ and $\ln 2 + 7i\pi/4$, and examples of z's which take these values can be found by looking at the argument. One such choice is that $f(1) = \ln 2 - i\pi/4$ and $f(-2-i) = \ln 2 + 7i\pi/4$.

9. Suppose $f(x, y) = x^2 + y^2 i$. Determine the set of z at which the Cauchy-Riemann equations are satisfied. Determine the set of z at which f is differentiable. Determine the set of z at which f is analytic.

Well, let $u(x, y) = x^2, v(x, y) = y^2$. These functions all satisfy the conditions of Theorem 3, p. 82, so differentiability at a point is equivalent to the Cauchy-Riemann equations being satisfied at a point. We have $u_x = 2x, u_y = 0, v_x = 0, v_y = 2y$. Thus, f is differentiable at $z_0 = x_0 + iy_0$ if and only if $2x_0 = 2y_0$ and $0 = -0$; that is, $y_0 = x_0$. So f is differentiable for points on a line, but it is analytic nowhere, because there is no point which contains an open set on which f is differentiable: the line contains no disk.

10a. Let γ denote the circle $|z| = 2$, traversed in a counter-clockwise fashion. Use the standard estimate for integrals; i.e., p.62(3), to show that $\left|\int_\gamma \frac{dz}{8+3\bar{z}}\right| \le 2\pi$. If $|z| = 2$, then $|8 + 3\bar{z}| \ge 8 - |3\bar{z}| = 8 - 3 \cdot 2 = 2$, hence $|\frac{1}{8+3\bar{z}}| \le \frac{1}{2}$. Thus, using the "standard estimate", since the length of γ is $2 \cdot 2\pi = 4\pi$,

$$\left|\int_\gamma \frac{dz}{8+3\bar{z}}\right| \le\ 4\pi \cdot \frac{1}{2} = 2\pi.$$

10b. By considering $|8 + 3\bar{z}|^2$ separately on the semicircles in the half-planes $x \ge 0$ and $x \le 0$, improve this estimate to $\left|\int_\gamma \frac{dz}{8+3\bar{z}}\right| \le \frac{6}{5}\pi$. More specifically, if $|z| = 2$, then $z = x+iy$, where $x^2+y^2 = 4$. Then $|8+3\bar{z}|^2 = (8+3x)^2+(-3y)^2 = 64+48x+9x^2+9y^2 = 100+48x$. If $x > 0$, then $|8 + 3\bar{z}|^2 \ge 100$, so $|8 + 3\bar{z}| \ge 10$ and $|\frac{1}{8+3\bar{z}}| \le \frac{1}{10}$. (If $x < 0$, then all you can say is that $48x \ge -96$, so that $|8 + 3\bar{z}|^2 \ge 4$, as before. Writing $\gamma = \gamma_1 \cup \gamma_2$, where γ_1 denotes the semicircle in the half-plane $x \ge 0$ and γ_2 is the other one. The length of each γ_j is 2π, and combining the estimates, we have

$$\left|\int_\gamma \frac{dz}{8+3\bar{z}}\right| = \left|\int_{\gamma_1} \frac{dz}{8+3\bar{z}} + \int_{\gamma_2} \frac{dz}{8+3\bar{z}}\right| \le \left|\int_{\gamma_1} \frac{dz}{8+3\bar{z}}\right| + \left|\int_{\gamma_2} \frac{dz}{8+3\bar{z}}\right| \le \frac{2\pi}{10} + \frac{2\pi}{2} = \frac{6\pi}{5}.$$

We will be able to show, before too long, that the exact value of the integral is $-\frac{3}{8}\pi i$.