**Math 348** **Homework 5 Solutions** **Due Wednesday, Feb. 28, 2001**

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1. §2.3 – 2. Compute using Cauchy's Formula or Theorem: $\int_{|z|=2} \frac{e^z}{z(z-3)}\, dz$. Here, and unless otherwise stated, we presume that the circle is traversed in a positive orientation. The integrand is analytic, except where the denominator vanishes, because it is the quotient of two entire functions. Thus, "problem points" are $z = 0$ and $z = 3$, of which only the first is inside the contour. Hence, if we let $f(z) = \frac{e^z}{z-3}$, then we can write the integral precisely in the form of Cauchy's Theorem:

$$\int_{|z|=2} \frac{e^z}{z(z-3)}\, dz = \int_{|z|=2} \frac{f(z)}{z-0}\, dz = 2\pi i \cdot f(0) = 2\pi i \cdot \tfrac{e^0}{0-3} = -\tfrac{2}{3}\pi i.$$

In case you are wondering what happens if both "problem points" are inside the contour, just wait until §2.6 and the residue theorem.

2. §2.3 – 4. Compute using Cauchy's Formula or Theorem $\int_{|z|=1} \frac{\sin z}{z}\, dz$. Even easier: Let $f(z) = \sin z$, which is entire. Since $z = z - 0$ and $0$ is inside $|z| = 1$, we have by Cauchy's Theorem that the value of the integral is $2\pi i \cdot \sin 0 = 0$. In fact, there's a better reason that this integral is 0. It turns out that, in some sense, $\frac{\sin z}{z}$ *is* analytic at $z = 0$, even though it's not defined there. Just wait until §2.5 and "removable singularities".

3. §2.3 – 10. Compute $\int_\gamma \left(z + \frac{1}{z}\right) dz$, where $\gamma$ is any curve in the upper half plane joining $-4+i$ to $6+2i$. The idea here is to find an antiderivative for $z + \frac{1}{z}$ that is valid in the upper half plane, and as we've seen, $\frac{1}{2}z^2$ is an (entire) antiderivative for $z$, and $Log(z)$, which is analytic in the upper half plane, is an antiderivative for $1/z$ in that domain. Thus, the value of the integral is $F(6+2i) - F(-4+i)$, where $F(z) = \frac{1}{2}z^2 + Log(z)$. This simplifies to

$$\tfrac{1}{2}(32+24i) + \ln\sqrt{40} + i\mathrm{Arg}(6+2i) - \left(\tfrac{1}{2}(15-8i) + \ln\sqrt{17} + i\mathrm{Arg}(-4+i)\right).$$

Further simplification is possible, but not edifying. If you do it numerically, remember that $-4+i$ is in the second quadrant, so its Argument should lie in $(\pi/2, \pi)$.

4. §2.4 – 2. Let $f(z) = (e^z - 1)^2$. Then $f(z) = 0$ if and only if $e^z = 1$; that is, if and only if $z = 2n\pi i$ for $n \in \mathbf{Z}$. The quickest way to do this problem is to write out $f$ and enough derivatives:

$$f(z) = e^{2z} - 2e^z + 1, \qquad f'(z) = 2e^{2z} - 2e^z, \qquad f''(z) = 4e^{2z} - 2e^z.$$

Since $e^{2z} = (e^z)^2$, if $e^{z_0} = 1$, then $f(z_0) = f'(z_0) = 0, f''(z_0) = 2$, hence $f$ has a zero of order two at each such $z_0$. There are valid and more sophisticated ways to do this problem.

5. §2.4 – 8. Same problem for $f(z) = \frac{z}{z^2+1}$. And remember that it's calling for the order of the zero. There is only one zero for $f$, at $z = 0$, and

$$f'(z) = \frac{(z^2+1)\cdot 1 - (2z)\cdot z}{(z^2+1)^2} = \frac{1-z^2}{(1+z^2)^2} \implies f'(0) = 1.$$

Thus $f$ has a zero of order 1 at $z = 0$.

6. Test 1 flashback: Find all possible values of the function $g(z) = \text{Arg}(z^3) - 3\text{Arg}(z)$, $z \neq 0$, where "Arg" denotes the Principal Value of the argument. Also, give a detailed description of the regions in the plane where $g$ takes its various values.

If $z = re^{it}$, $-\pi \leq t < \pi$, then $\text{Arg}(z) = t$, and $z^3 = r^3 e^{3it}$, so $\text{Arg}(z^3) = 3t + 2\pi n$, where $n$ is the unique integer that makes $3t + 2\pi n \in [-\pi, \pi)$. There are 3 cases:
i. If $t \in [-\pi/3, \pi/3)$, then $3t \in [-\pi, \pi)$, so $n = 0$ and $g(z) = 3t - 3t = 0$.
ii. If $t \in [\pi/3, \pi)$, then $3t \in [\pi, 3\pi)$, so $n = -1$ and $g(z) = 3t - 2\pi - 3t = -2\pi$.
iii. If $t \in [-\pi, -\pi/3)$, then $3t \in [-3\pi, -\pi)$, so $n = 1$ and $g(z) = 3t + 2\pi - 3t = 2\pi$.

Don't let the similarity of the answer here to the exam fool you, if "3" were replaced by "4", or "5", there would be more cases.

7. Test 1 flashback: Find a polynomial $h(z)$ so that $\sum_{n=1}^{\infty} n^3 z^n = \frac{h(z)}{(1-z)^4}$ for $|z| < 1$.

As a general rule: if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is convergent for $|z| < r$, then

$$zf'(z) = z \sum_{n=0}^{\infty} a_n (nz^{n-1}) = \sum_{n=0}^{\infty} n a_n z^n, \qquad \text{for } |z| < r.$$

Applying this to $f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$, we get the earlier homework problem:

$$\sum_{n=1}^{\infty} n z^n = z \left( \frac{1}{1-z} \right)' = \frac{z}{(1-z)^2}.$$

(Note that the term for $n = 0$ vanishes, so I'll just drop it from the summations.) Repeating this principle one more time, we get the problem from the exam:

$$\sum_{n=1}^{\infty} n^2 z^n = z \left( \frac{z}{(1-z)^2} \right)' = z \left( \frac{1}{(1-z)^2} + \frac{z(-2)(-1)}{(1-z)^3} \right) = \frac{z(1-z+2z)}{(1-z)^3} = \frac{z+z^2}{(1-z)^3}.$$

(The use of the product rule, rather than the quotient rule, is just a personal preference.) Finally, we do it one more time, to get $h(z) = z + 4z^2 + z^3$:

$$\begin{aligned} \sum_{n=1}^{\infty} n^3 z^n &= z \left( \frac{z+z^2}{(1-z)^3} \right)' = z \left( \frac{1+2z}{(1-z)^3} + \frac{(z+z^2)(-3)(-1)}{(1-z)^4} \right) \\ &= \frac{z(1+2z)(1-z) + 3z^2(1+z))}{(1-z)^3} = \frac{z+4z^2+z^3}{(1-z)^4}. \end{aligned}$$

8. ($\mathcal{E}$) Use Cauchy's formula to evaluate $\int_{|z|=1} \frac{dz}{(2z-1)(z-2)}$. Hint: be careful about factors of 2. Then substitute $z = e^{i\theta}$ into your answer to obtain a formula of the kind: $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = c$. Your job is to find $a$, $b$ and $c$.

Here the denominator vanishes at $z = \frac{1}{2}, 2$, and only the former is within the circle of integration. To put this in the form of Cauchy's formula, write

$$\frac{1}{(2z-1)(z-2)} = \frac{\frac{1}{2(z-2)}}{z-\frac{1}{2}}.$$

(This is what I meant about factors of 2.) Then, by Cauchy's Formula, we have

$$\int_{|z|=1} \frac{dz}{(2z-1)(z-2)} = 2\pi i \frac{1}{2(\frac{1}{2}-2)} = -\tfrac{2\pi i}{3}.$$

If we now make the substitution $z = e^{i\theta}$, $0 \le \theta \le 2\pi$, into the integral, we have

$$\begin{aligned}\int_{|z|=1} \frac{dz}{(2z-1)(z-2)} &= \int_0^{2\pi} \frac{ie^{i\theta}\,d\theta}{2e^{2i\theta}-5e^{i\theta}+2} \\ = i\int_0^{2\pi} \frac{d\theta}{2(e^{i\theta}+e^{-i\theta})-5} &= i\int_0^{2\pi} \frac{d\theta}{4\cos\theta-5}.\end{aligned}$$

We therefore have one of several possible formulas:

$$\int_0^{2\pi} \frac{d\theta}{-5+4\cos\theta} = -\frac{2\pi}{3}, \qquad \int_0^{2\pi} \frac{d\theta}{5-4\cos\theta} = \frac{2\pi}{3}, \qquad \text{etc.}$$

9. Let $D$ consist of the complex plane minus the negative real axis. Define $f$, a "branch" of $z^{1/2}$ on $D$, by

$$f(re^{it}) = r^{1/2}e^{it/2}, \quad t \in (-\pi, \pi).$$

Find an explicit antiderivative for $f$ in $D$ using the method of section 2.3, using $z_0 = 1$ as a "basepoint". Hint: take as your contour from $z_0$ to $z = re^{it}$ as the arc from 1 to $e^{it}$ followed by the ray from $e^{it}$ to $re^{it}$.

I did a similar problem in class on 2/23. Following the instructions,

$$F(z) = \int_{\tau=0}^{t} f(e^{i\tau})d(e^{i\tau}) + \int_{\rho=1}^{r} f(\rho e^{it})\,d(\rho e^{it}).$$

Here, I have chosen variables $\tau$ and $\rho$, because I gave the confusing names of $t$ and $r$ for the specific destination of the integral. On the contours, we have $f(e^{i\tau}) = e^{i\tau/2}$ (because $r = 1$) and $f(\rho e^{it}) = \rho^{1/2} \cdot e^{it/2}$, hence,

$$\begin{aligned}\int_{\tau=0}^{t} f(e^{i\tau})\,d(e^{i\tau}) &= \int_{\tau=0}^{t} e^{i\tau/2} i e^{i\tau}\,d\tau = \int_{\tau=0}^{t} ie^{3i\tau/2}\,d\tau \\ &= \tfrac{2}{3}e^{3i\tau/2}\Big|_{\tau=0}^{t} = \tfrac{2}{3}\left(e^{3it/2} - e^{i\cdot 0}\right).\end{aligned}$$

and

$$\int_{\rho=1}^{r} f(\rho e^{it})\ d(\rho e^{it}) = \int_{\rho=1}^{r} \rho^{1/2} e^{it/2} \left(e^{it}\ d\rho\right)$$
$$= e^{3it/2} \int_{\rho=1}^{r} \rho^{1/2}\ d\rho = e^{3it/2} \cdot \tfrac{2}{3}\rho^{3/2}\Big|_{\rho=1}^{r} = \tfrac{2}{3}e^{3it/2}(r^{3/2}-1).$$

And, therefore,

$$F(z) = \tfrac{2}{3}\left(e^{3it/2} - e^{i\cdot 0} + e^{3it/2}(r^{3/2}-1)\right) = \tfrac{2}{3}r^{3/2}e^{3it/2} - \tfrac{2}{3}.$$

Up to a constant, $F(z) = \frac{2}{3}e^{3\mathrm{Log}z/2}$, which is what we might expect.

10a. Let $w$ be a fixed, but otherwise unspecified complex number. Determine all complex numbers $z$ with the property that $\cos(z) = \cos(w)$. Hint: when you get to the quadratic in $e^{iz}$, you will be able to factor it into two linear factors.

We start as we usually do, without simplifying $\cos w$:

$$e^{iz} + e^{-iz} = 2\cos z = 2\cos w = e^{iw} + e^{-iw} \implies e^{2iz} - (e^{iw} + e^{-iw})e^{iz} + 1 = 0$$
$$\implies (e^{iz} - e^{iw})(e^{iz} - e^{-iw}) = 0.$$

Thus, either $e^{iz} = e^{iw}$ or $e^{iz} = e^{-iw}$. Taking logarithms, and keeping in mind the additive term in the lgos, we have

$$e^{iz} = e^{iw} \implies iz = iw + 2\pi in \implies z = w + 2\pi n, \qquad n \in \mathbf{Z},$$

and

$$e^{iz} = e^{-iw} \implies iz = -iw + 2\pi in \implies z = -w + 2\pi n, \qquad n \in \mathbf{Z}.$$

Thus, $z = \pm w + 2\pi n$ for $n \in \mathbf{Z}$, just as in the real case.

10b. Let $w$ be a fixed, but otherwise unspecified complex number. Determine all complex numbers $z$ with the property that $\sin(z) = \sin(w)$. Hint: when you get to the quadratic in $e^{iz}$, you will be able to factor it into two linear factors.

Applying the same arguments, we have

$$e^{iz} - e^{-iz} = 2i\sin z = 2i\sin w = e^{iw} - e^{-iw} \implies e^{2iz} - (e^{iw} - e^{-iw})e^{iz} - 1 = 0$$
$$\implies (e^{iz} - e^{iw})(e^{iz} + e^{-iw}) = 0.$$

In the first case, we already know that $z = w + 2\pi n$ for some $n \in \mathbf{Z}$. The second is just a little trickier; by absorbing $-1 = e^{i\pi}$ into the second term, we have

$$e^{iz} = -e^{-iw} \implies e^{iz} = e^{i\pi - iw} \implies iz = i\pi - iw + 2\pi in \implies z = \pi - w + 2\pi n, \qquad n \in \mathbf{Z}.$$

Thus, either $z = w + 2\pi n$ or $z = \pi - w + 2\pi n$, as in the real case.