Math 348 Homework 6 Solutions Due Monday, Mar. 19, 2001

1. §2.4 – 10. "Find the power-series expansion about the given point for each of the functions; find the largest disc in which the series is valid": here, e^z about $z_0 = \pi i$. If $f(z) = e^z$, then $f^{(n)}(z) = e^z$ for all $n \in \mathbf{N}$. Since the fundamental formula is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n,$$

and $e^{\pi i} = -1$, we obtain the odd-looking formula:

$$e^z = \sum_{n=0}^{\infty} \frac{-1}{n!}(z - \pi i)^n.$$

(This isn't so strange if you think about it: the right hand side is simply $-e^{z-\pi i}$.) Since f is entire, it is analytic on disks of the form $|z - \pi i| < R$ for every R. Thus, the power series converges on each such disk, and therefore converges for all z.

2. §2.4 – 12. Same question, for $\frac{z^2}{1-z}$ at $z_0 = 0$. There are a lot of ways to do this problem. The fastest is to multiply the respective power series for z^2 at $z_0 = 0$ and $\frac{1}{1-z}$ at $z_0 = 0$:

$$(z^2) * \sum_{n=0}^{\infty} z^n = \sum_{n=2}^{\infty} z^n = z^2 + z^3 + \ldots$$

Another way of doing this is to divide the numerator into the denominator:

$$\frac{z^2}{1-z} = \frac{(z^2 - 1) + 1}{1 - z} = -1 - z + \frac{1}{1-z}.$$

The first two terms of $\frac{1}{1-z}$ are erased by $-1 - z$ as before. The series converges precisely for $|z| < 1$, as we already know, and this is the largest possible disk.

3. §2.4 – 20. Suppose f is entire and $Re(f(z)) < c$ for all z. Following the hint, and writing $g(z) = e^{f(z)}$, we note that g is entire and $|g(z)| = e^{Re(f(z))} < e^c$. It then follows by Liouville's Theorem that g is constant, say $g(z) = k \neq 0$. A small argument is necessary now to show that f is constant: we know that $e^{f(z)} = k$. Thus, for all z, $f(z) = \mathrm{Log}(k) + 2\pi i n(z)$ for some integer $n(z)$ which *might* depend on z. However, f is continuous (since it is entire), and this means that $n(z)$ is continuous. And a continuous function which takes only integer values is constant, so f is constant. (If you don't know the proof of this last remark, think about the definition of continuity with $\epsilon = 1/3$.)

4. §2.4 – 27 and 28a (note that the solution to 27 is essentially given in the back.) Suppose f and A are analytic in a simply-connected domain D and $f'(z) = A(z)f(z)$ for $z \in D$. You are asked to prove that

$$f(z) = Ce^{\int_{z_0}^{z} A(w)\, dw}$$

for some constant C, where the integral is taken over any piecewise smooth path connecting a basepoint z_0 to z. Again following the hint, let

$$g(z) = e^{-\int_{z_0}^{z} A(w)\ dw}.$$

Since A is analytic in a simply-connected domain, the function $\int_{z_0}^{z} A(w)\ dw$ is well-defined and analytic in D, and an antiderivative for $A(z)$. Thus, by the chain rule, $g'(z) = g(z)(-A(z))$, and so,

$$(f(z)g(z))' = f(z)g'(z) + f'(z)g(z) = -A(z)f(z)g(z) + A(z)f(z)g(z) = 0.$$

This means that $f(z)g(z)$ is constant on D; using $g(z_0) = e^0$, we have

$$f(z)g(z) = f(z_0)g(z_0) = f(z_0) \implies f(z) = \frac{f(z_0)}{g(z)} = f(z_0)e^{\int_{z_0}^{z} A(w)\ dw},$$

as desired. With $A(z) = -2z$, we can take $D = \mathbf{C}$, and for simplicity, $z_0 = 0$. Thus,

$$f(z) = f(0)e^{\int_0^z 2w\ dw} = f(0)e^{z^2}.$$

5. ($\mathcal{E}$) Evaluate two integrals:

$$\frac{1}{2\pi i}\int_{|z|=1}\left(z + \frac{2}{z}\right)^3\ dz = \frac{1}{2\pi i}\int_{|z|=1}\left(z^3 + 6z + \frac{12}{z} + \frac{8}{z^3}\right)\ dz = 12,$$

since the integral of z^n is 0 if $n \neq -1$ and 1 if $n = 1$. And

$$\frac{1}{2\pi i}\int_{|z|=2}\frac{dz}{z^2 - 3z} = \frac{1}{2\pi i}\int_{|z|=2}\frac{\frac{1}{z-3}\ dz}{z - 0} = \frac{1}{0-3} = -\frac{1}{3},$$

since 0 is inside $|z| = 2$ and $\frac{1}{z-3}$ is analytic there. (Or use partial fractions.)

6. ($\mathcal{E}$) Evaluate the following integrals, where C denotes the contour $|z| = 2$, taken in the usual counterclockwise way:

$$\frac{1}{2\pi i}\int_C \frac{\cos z}{z}\ dz = \cos 0 = 1$$

by the usual procedure, since $\cos z$ is entire, 0 is inside C and $z = z - 0$;

$$\frac{1}{2\pi i}\int_C \frac{e^{3z}}{z^4}\ dz = \frac{1}{3!}(e^{3z})'''\big|_{z=0} = \frac{3^3 e^{3z}}{6}\big|_{z=0} = \frac{9}{2}$$

by Cauchy's Theorem, since e^{3z} is entire and $z^4 = (z-0)^4$;

$$\frac{1}{2\pi i}\int_C e^{3z}(z-1)^4\ dz = 0,$$

because the integrand is entire.

7. Evaluate the following integrals, where C denotes the contour $|z| = 1$, taken in the usual counterclockwise way:

$$\frac{1}{2\pi i}\int_C \frac{z}{e^z}\,dz = 0,$$

because the integrand is entire;

$$\frac{1}{2\pi i}\int_C \frac{e^{3z}}{(z-3)^4}\,dz = 0,$$

because the integrand, ze^{-z}, is analytic in $|z| < 5/2$ (for example), which contains C and its interior; finally, since $-\frac{3}{4}$ lies within C,

$$\frac{1}{2\pi i}\int_C \frac{1}{3+4z}\,dz = \frac{1}{4}\left(\frac{1}{2\pi i}\int_C \frac{1}{z-(-\frac{3}{4})}\,dz\right) = \frac{1}{4}.$$

8. ($\mathcal{E}$) Let

$$f(z) = \frac{1}{1-2z} + \frac{1}{1+z}.$$

Write down the power series for f centered at $z_0 = 0$ and and $z_0 = 3$. This can be done in at least two different ways: either by computing $f^{(n)}(z)$ by an easy induction and evaluation at z_0, or by manipulating the geometric series.

a. We have already seen that if $g(z) = (1-z)^{-1}$, then $g^{(n)}(z) = n!(1-z)^{-(n+1)}$. It follows from the chain rule that if $h(z) = (1-az)^{-1}$, then $h^{(n)}(z) = n!a^n(1-az)^{-(n+1)}$. Thus,

$$f^{(n)}(z) = n!\left(\frac{2^n}{(1-2z)^{n+1}} + \frac{(-1)^n}{(1+z)^{n+1}}\right).$$

It follows that for $f^{(n)}(0) = n!(2^n + (-1)^n)$, and the power series at 0 is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(z-0)^n = \sum_{n=0}^{\infty}(2^n + (-1)^n)z^n.$$

This can also be found by adding the two basic geometric series.

b. Similarly,

$$f^{(n)}(3) = n!\left(\frac{2^n}{(-5)^{n+1}} + \frac{(-1)^n}{4^{n+1}}\right),$$

so that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(z-3)^n = \sum_{n=0}^{\infty}\left(-\frac{1}{5}\left(-\frac{2}{5}\right)^n 2^n + \frac{1}{4}\left(-\frac{1}{4}\right)^n\right)(z-3)^n.$$

The way to find this without differentiating is to rewrite f as a geometric series in $z-3$:

$$f(z) = \frac{1}{1-2(z-3+3)} + \frac{1}{1+(z-3+3)} = \frac{1}{-5-2(z-3)} + \frac{1}{4+z-3}$$
$$= -\frac{1}{5}\cdot\frac{1}{1+\frac{2}{5}(z-3)} + \frac{1}{4}\cdot\frac{1}{1+\frac{1}{4}(z-3)}.$$

9a. ($\mathcal{E}$) Suppose C is a piecewise smooth, (not necessarily closed!) contour. Prove that $\int_C z\,dz = 0$ implies $\int_C z^3\,dz = 0$. Let C have beginning point z_0 and endpoint z_1. Then we've known for some time that

$$\int_C z^n\,dz = \frac{1}{n+1}\left(z_1^{n+1} - z_0^{n+1}\right).$$

If $\int_C z\,dz = 0$, then $z_1^2 - z_0^2 = 0$, so $z_0^2 = z_1^2$, hence $z_0^4 = z_1^4$, so that $\int_C z^3\,dz = 0$.
9b. Find a simple, piecewise smooth contour C so that $\int_C z^3\,dz = 0$ and $\int_C z\,dz = 1$. That would be a contour from z_0 to z_1 so that

$$0 = \tfrac{1}{4}(z_1^4 - z_0^4), \qquad 1 = \tfrac{1}{2}(z_1^2 - z_0^2)$$

That is, $z_1^4 = z_0^4$ and $z_1^2 - z_0^2 = 2$. Since $z_1^4 = z_0^4$, we have $z_1^2 = \pm z_0^2$, and, when combined with the second equation, we must have $z_1^2 = 1$, $z_0^2 = -1$. Therefore, any contour which begins at i or $-i$ and ends at 1 or -1 will work.

10. Let C denote a contour consisting of a line segment from $1-2i$ to $4i$ followed by a line segment from $4i$ to $2+i$. Define a branch of the logarithm which is analytic on a domain containing C and use it to evaluate $\int_C \frac{dz}{z}$. This problem requires both a number and a function.

There was a typo in the original, and you can actually use the Principal Value of the logarithm here. (What I *meant* will be on homework 7.). Since the line segments do not cross the negative real axis, we just wind up with $\mathrm{Log}(2+i) - \mathrm{Log}(1-2i)$; that is,

$$\ln(\sqrt{5}) + i\arctan(1/2) - \ln(\sqrt{5}) - i\arctan(-2) = \frac{i\pi}{2}.$$