Math 348 Homework 7 Solutions Due Wednesday, March 28, 2001

1. §2.5 – 2. Let $f(z) = \frac{z^2}{\sin z}$. Since the numerator and denominator are entire, the singularities of f occur where the denominator vanishes: at $z = n\pi$. We need to distinguish two cases: if $z_n = n\pi$, $n \neq 0$, then $z_n \neq 0$, $\sin z_n = 0$, $\cos z_n (= (-1)^n) \neq 0$ means that the numerator is not zero and the denominator has a zero of order 1 at z_n. Thus, f has a pole of order one at z_n, $n \neq 0$. In the remaining case, $z_0 = 0$, the numerator has a zero of order 2, hence f has a removable singularity, with a zero of order $2 - 1 = 1$. Thus, the singularity can be removed by giving f the value 0. The power series at z_0, with assistance from Mathematica, is

$$\frac{z^2}{\sin z} = \frac{z^2}{z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \dots} = z + \frac{z^3}{6} + \frac{7z^5}{360} + \frac{31z^7}{15120} + \frac{127z^9}{604800} + \frac{73z^{11}}{3421440} + \dots$$

2. §2.5 – 6. Let $f(z) = \frac{e^z - 1}{e^{2z} - 1}$. Again, this is a quotient of two entire functions, so the only singularities arise when the denominator vanishes; that is, when $e^{2z} = 1$, or $2z = 2n\pi i$, or $z = z_n = n\pi i$ for $n \in \mathbf{N}$. If $(e^z)^2 = 1$, then $e^z = -1$ or $e^z = 1$, corresponding to z_n for odd n and even n, respectively. Since the derivative of the denominator is $2e^{2z}$, which equals 2 at every z_n, it follows that the denominator has a zero of order 1 at each z_n. The numerator equals $-1 - 1 = -2$ when n is odd, so f has a pole of order 1 at $z_{2k+1} = (2k+1)\pi i$. The numerator equals $1 - 1 = 0$ when n is even, but its derivative is equal to $e^z \neq 0$, so f has a removable singularity at $z_{2k} = 2k\pi i$. In order to find the value, it's ok to make a computational trick: if $e^z - 1 \neq 0$, then

$$f(z) = \frac{e^z - 1}{e^{2z} - 1} = \frac{e^z - 1}{(e^z + 1)(e^z - 1)} = \frac{1}{e^z + 1}.$$

We see that $\lim_{z \to 2k\pi i} f(z) = \frac{1}{2}$.

3. §2.5 – 8. Find the Laurent series for $f(z) = \frac{z^2}{z^2 - 1}$ at $z_0 = 1$, and give the residue. The fastest way I can see to do this uses partial fractions:

$$\frac{z^2}{z^2 - 1} = 1 + \frac{1}{z^2 - 1} = 1 + \frac{\frac{1}{2}}{z - 1} - \frac{\frac{1}{2}}{z + 1}.$$

The first and second terms above already fit into the pattern of the Laurent series at $z_0 = 1$, and we can write the third as a power series in $z - 1$:

$$-\frac{\frac{1}{2}}{z + 1} = -\frac{\frac{1}{2}}{2 + (z - 1)} = -\frac{1}{4} \cdot \frac{1}{1 + \frac{1}{2}(z - 1)} = -\frac{1}{4} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} (z - 1)^n.$$

Putting it all together, and combining the two constant terms, we have

$$f(z) = \frac{1}{2} \cdot \frac{1}{z - 1} + \frac{3}{4} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{n+2}} (z - 1)^n.$$

The residue at 1 is $\frac{1}{2}$, and this can also be found by one of the "techniques":

$$\lim_{z\to 1}\left((z-1)\frac{z^2}{z^2-1}\right)=\lim_{z\to 1}\left(\frac{z^2}{z+1}\right)=\frac{1}{2}.$$

I will report to the class if I find any other easy methods on your papers.

4. §2.5 – 14. Suppose f is analytic in $|z-z_0|<R$ and has a zero of order m at z_0. Show that $Res(\frac{f'}{f};z_0)=m$. By hypothesis, $f(z)=(z-z_0)^m h(z)$, where h is analytic in $|z-z_0|<R$ and $h(z_0)\neq 0$. Since h is analytic, there is a disk $|z-z_0|<r(\leq R)$ so that $h(z)\neq 0$. Thus, by the product rule, we have

$$\frac{f'(z)}{f(z)}=\frac{(z-z_0)^m h'(z)+m(z-z_0)^{m-1}h(z)}{(z-z_0)^m h(z)}=\frac{m}{z-z_0}+\frac{h'(z)}{h(z)}.$$

Since $h\neq 0$ in a neighborhood of z_0, $\frac{h'}{h}$ is analytic there, and it follows that $\frac{f'}{f}$ has a pole of order 1 at $z=z_0$. The residue can be found in one of two ways: since $\frac{h'}{h}$ is analytic, it has a representation as a power series, so that the coefficient of $(z-z_0)^{-1}$ in $\frac{f'}{f}$ is clearly m, or by one of the computational techniques,

$$\lim_{z\to z_0}\left((z-z_0)\frac{f'(z)}{f(z)}\right)=\lim_{z\to z_0}\left(m+(z-z_0)\frac{h'(z)}{h(z)}\right)=m.$$

5. ($\mathcal{E}$) Let C denote the circle $|z-i|=1$, taken in the usual counterclockwise orientation. Compute by any correct method $\int_C \frac{z}{(z^2+1)^2}\,dz$.

The integrand, being a quotient of two polynomials, has singularities only where the denominator vanishes: at $z=i$ and $z=-i$, and of these, only i lives in $|z-i|<1$. Thus, the value of the integral is $2\pi i$ Res $\left(\frac{z}{(z^2+1)^2};i\right)$. The easiest way to compute the residue is to write

$$\frac{z}{(z^2+1)^2}=\frac{\left(\frac{z}{(z+i)^2}\right)}{(z-i)^2}:=\frac{g(z)}{(z-i)^2},$$

so that the residue is $\frac{1}{1!}g'(i)$. Writing $g(z)=z(z+i)^{-2}$, we see that $g'(z)=(z+i)^{-2}-2z(z+i)^{-3}$, so that $g'(i)=\frac{1}{(2i)^2}-\frac{2i}{(2i)^3}=0$, and the value of the integral is zero.

6. ($\mathcal{E}$) Find the Taylor series for $f(z)=\frac{1+2z}{(1-z)^2}$ at $z=0$. (Hint: either of the following identities might be useful: $1+2z=1+2\cdot z$; $1+2z=3+2(z-1)$.) The two basic series you know and/or remember are

$$\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n;\qquad \frac{1}{(1-z)^2}=\sum_{n=0}^{\infty}(n+1)z^n.$$

Using these hints respectively, we have

$$\frac{1+2z}{(1-z)^2} = \frac{1}{(1-z)^2} + \frac{2z}{(1-z)^2} = \sum_{n=0}^{\infty}(n+1)z^n + 2z\sum_{n=0}^{\infty}(n+1)z^n$$
$$= \sum_{n=0}^{\infty}(n+1)z^n + 2\sum_{n=0}^{\infty}nz^n = \sum_{n=0}^{\infty}(3n+1)z^n$$

or

$$\frac{1+2z}{(1-z)^2} = \frac{3}{(1-z)^2} + \frac{2(z-1)}{(1-z)^2} = \sum_{n=0}^{\infty}3(n+1)z^n - \frac{2}{1-z}$$
$$= \sum_{n=0}^{\infty}3(n+1)z^n + \sum_{n=0}^{\infty}(-2)z^n = \sum_{n=0}^{\infty}(3n+1)z^n.$$

7. ($\mathcal{E}$) Classify the singularity of

$$f(z) = \frac{\sin(z^3) - z^3}{z^{16}}$$

at $z = 0$ as one of {removable singularity, essential singularity, pole of order m for specific m }, and compute the residue of f at $z = 0$. Writing out the power series for the numerator, we see that

$$\sin(z^3) - z^3 = \left(z^3 - \frac{1}{3!}(z^3)^3 + \frac{1}{5!}(z^3)^5 - + \dots\right) - z^3 = -\frac{z^9}{3!} + \frac{z^{15}}{5!} - + \dots$$

Thus, the series for $f(z)$ at $z = 0$ is

$$f(z) = -\frac{z^{-7}}{3!} + \frac{z^{-1}}{5!} - + \dots,$$

hence f has a pole of order 7, with residue of $\frac{1}{120}$.

8a. ($\mathcal{E}$) Suppose f and g are entire functions and neither is identically zero. Suppose further that, for all z, $|f(z)| \leq |g(z)|$. Show that the only singularities of $h = \frac{f}{g}$ are removable ones at the zeros of g. (Hint: you know something about h in a neighborhood of a zero of g.) As we've noted earlier, the only singularities of h can be zeros of g, which are isolated (since g is entire, its zeros have no accumulation points). Further, $|h(z)| \leq 1$ for all z except at the zeros of g, so h is bounded at its isolated singularities. This means that each singularity is removable.

b. Prove that there is a constant c, $|c| \leq 1$ so that $f(z) = cg(z)$ for all z. By defining $H(z)$ to equal $h(z)$, when h is defined, and the "missing" value at each singularity, we see that H is entire and $|H(z)| \leq 1$ if z is not one of the singularities. By continuity, $|H(z)| \leq 1$ for all z, so by Liouville's Theorem, $H(z) = c$ is constant, with $|c| \leq 1$. Thus, if $g(z) \neq 0$, then $\frac{f(z)}{g(z)} = h(z) = c$. On the other hand, if $g(z) = 0$, then $|f(z)| \leq |g(z)|$ implies $f(z) = 0$ as well, and so for all z, $f(z) = cg(z)$.

9. ($\mathcal{E}$) Suppose $f(x+iy) = u(x,y) + iv(x,y)$ is entire, and $|348u(x,y) + 2001v(x,y)| < 1$ for all (x,y). Prove that f is constant. Let $g(z) = (348 - 2000i)f(z)$ so that

$$g(x+iy) = 348u(x,y) + 2001v(x,y) + i(-2001u(x,y) + 348v(x,y)).$$

Then $|Re(g(z)| \leq 1$, so that $Re(g(z)) \leq 1$, and by HW6#3, g is a constant, hence $f = (348 - 2000i)^{-1}g$ is also a constant.

10. Let C denote a contour consisting of a line segment from $1 - 2i$ to $4i$ followed by a line segment from $4i$ to $-2 - i$. Define a branch of the logarithm which is analytic on a domain containing C and use it to evaluate $\int_C \frac{dz}{z}$. This problem requires both a number and a function. As the sketch shows,

we cannot use the Principal Value of the logarithm, since its branch cut, the negative real axis, crosses the contour. We can use the indicated branch cut. More formally, we define

$$\log z = \ln|z| + i \arg(z), \qquad -\tfrac{\pi}{2} < \arg(z) < \tfrac{3\pi}{2}.$$

This is a function whose derivative is $1/z$, and which is analytic in a domain containing C. Thus, the value of the integral is

$$\begin{aligned} &\log(-2-i) - \log(1-2i) \\ &= \ln\sqrt{5} + i\arg(-2-i) - (\ln\sqrt{5} + i\arg(1-2i)) \\ &= i\big(\arg(-2-i) - \arg(1-2i)\big). \end{aligned}$$

We see from the picture that $\arg(-2-i) - \arg(1-2i) \in (0, 2\pi)$, and since $-2 - i = -i(1-2i)$, we conclude that

$$\int_C \frac{dz}{z} = \frac{3\pi i}{2}.$$