Math 348 **First Examination** **February 19, 2001**

Instructions **Name** ______________________________

1. This is a closed book, closed notes, non-collaborative exam.

2. There are 6 questions on 6 pages: point values vary, need not reflect difficulty, and sum to 100. Different parts of the same problem might (or might not) be related. You may quote theorems from the book, the class or the homework, provided you do so correctly.

3. **Read the problems carefully.** Partial credit will be given when earned. Complete sentences are not mandatory under test conditions. Please indicate if you make meaningful use of the back of the sheets (or the sheets at the end) as scratch paper.

4. **Read the problems carefully.**

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1. (15 points) Let C be the semi-circular contour from $-2i$ to $2i$ along the circle of radius 2 taken counterclockwise. Compute by any correct method

$$\int_C \left(z^2 + \bar{z}\right)\ dz.$$

2. For each of the following equations, find all complex solutions z.

2a. (6 points) $z^4 = -1$.

2b. (7 points) $e^z = 2 + 2i$.

2c. (7 points) $\cos z = 3$.

3. (15 points) Find all possible values of the function $g(z) = \mathrm{Arg}(z^2) - 2\mathrm{Arg}(z)$, $z \neq 0$, where "Arg" denotes the Principal Value of the argument. It is not necessary in this problem to write a detailed description of the regions in the plane where g takes its various values.

4a. (10 points) Give an example of an open set that isn't connected, and an example of a connected set that isn't open. It suffices to draw two carefully labeled pictures.

4b. (10 points) Because you studied for it: determine all values of $(2i)^i$.

5a. (15 points) Find a complex number $\alpha = \rho e^{it}$ and a positive integer n so that the image of region A under the map $w = \alpha z^n$ is the region B. (Hint: first look at what happens to $z = re^{i\theta}$ under this map.) The region A is the pizza-slice-shaped region in the first quadrant, bounded by the lines $x = y$ and the positive imaginary axis, and in the disk $|z| \leq 2$. The region B is a wider set in the upper half w-plane, bounded by the positive real axis and the line $u + v = 0$ and within the disk $|w| \leq 2$.

5b. (10 points) We have seen that, for $|z| < 1$.

$$\frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots = \sum_{n=1}^{\infty} nz^n.$$

Use this fact, and what we know about differentiating power series, to find a polynomial $h(z)$ so that

$$g(z) = \sum_{n=1}^{\infty} n^2 z^n = \frac{h(z)}{(1-z)^3} \qquad \text{for } |z| < 1.$$

6. (5 points) Poor Elmer Ph.D.! He is brilliant at computing, but he always calculates before he starts to think, and it gets him into trouble. Here's his problem:

"I want to find an analytic function f whose real part is $u(x, y) = x^2 + y^2$, so I want to use the Cauchy-Riemann equations to find v, but I keep getting stuck. I know that

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -2y,$$

so that $v(x, y) = -2xy + h(y)$ for some function h which depends on y. But then

$$\frac{\partial v}{\partial y} = -2x + h'(y) = \frac{\partial u}{\partial x} = 2x,$$

so that $h'(y) = 4x$. But that doesn't look right to me. I can't find a single thing wrong in these calculations. HELP!"

Elmer is right: there's nothing wrong in his calculations. But what is his underlying mistake?