

Notes on Quadrature and Bonus Honor Questions – Math 242, 11/12/01

A *quadrature formula* estimates an integral by a finite sum:

$$\int_a^b f(x) dx \approx \sum_{k=1}^n \lambda_k f(x_k). \quad (1)$$

The simplest quadrature formula is of course the Riemann sum. More sophisticated versions include the trapezoid rule and Simpson's Law. (See §7.7 of the text.)

A quadrature formula (1) is said to have *strength* k if

$$\int_a^b f(x) dx = \sum_{k=1}^n \lambda_k f(x_k)$$

for every polynomial f with degree $\leq k$. For example, (1) has strength 0 if and only if

$$\int_a^b c dx = \sum_{k=1}^n \lambda_k \cdot c \iff b - a = \sum_{k=1}^n \lambda_k.$$

It is worth noting that there is no quadrature formula which can work for all polynomials. In fact, suppose (1) is given, and let $f(x) = \prod(x - x_k)^2$. Then $f(x) \geq 0$ for all x and it's easy to see that $\int_a^b f(x) dx > 0$, but $f(x_k) = 0$ for all k , so the sum in (*) is zero, regardless of the λ_k 's.

For simplicity, I will assume that $a = 0$ and $b = 1$, and look for exact quadrature formulas of the type

$$\int_0^1 f(t) dt = \sum_{k=1}^n \lambda_k f(t_k). \quad (2)$$

To move between (1) and (2), simply let $x = a + (b - a)t$. The coefficients λ_k are multiplied by $b - a$ in going from (2) to (1).

The first interesting case is to look at quadrature formulas of strength 2 on $[0, 1]$. A necessary and sufficient condition is that, for all (α, β, γ) , we have

$$\int_0^1 (\alpha + \beta t + \gamma t^2) dt = \sum_{k=1}^n \lambda_k (\alpha + \beta t_k + \gamma t_k^2),$$

that is,

$$\alpha + \frac{\beta}{2} + \frac{\gamma}{3} = \alpha \sum_{k=1}^n \lambda_k + \beta \sum_{k=1}^n \lambda_k t_k + \gamma \sum_{k=1}^n \lambda_k t_k^2.$$

It's easy to see that this holds for all (α, β, γ) if and only if

$$1 = \sum_{k=1}^n \lambda_k, \quad \frac{1}{2} = \sum_{k=1}^n \lambda_k t_k, \quad \frac{1}{3} = \sum_{k=1}^n \lambda_k t_k^2. \quad (3)$$

This is a system of three equations in $2n$ unknowns, the λ_k 's and the t_k 's.

The simplest way to solve this is to take $t_1 = 0$, $t_2 = \frac{1}{2}$ and $t_3 = 1$, leaving three *linear* equations in the three λ_k 's:

$$\lambda_1 + \lambda_2 + \lambda_3 = 1, \quad \frac{1}{2}\lambda_2 + \lambda_3 = \frac{1}{2}, \quad \frac{1}{4}\lambda_2 + \lambda_3 = \frac{1}{3}. \quad (4)$$

It's not hard to solve (4) and obtain a quadrature formula of strength 2 that you recognize from Simpson's Rule:

$$\int_0^1 f(t) dt = \frac{1}{6} (f(0) + 4f(\frac{1}{2}) + f(1)).$$

After making the linear change described earlier, this becomes

$$\int_a^b f(x) dx = \frac{b-a}{6} (f(a) + 4f(\frac{a+b}{2}) + f(b)).$$

If an object is such that its cross-sectional area is a quadratic polynomial in its height, and it has height H , top area T , middle area M and bottom area B , then its volume is $\frac{H}{6}(T + 4M + B)$. For example, a sphere of radius r has $H = 2r$, $T = B = 0$ and $M = \pi r^2$; a right circular cone with base r and height H has $T = 0$, $M = \pi \left(\frac{r}{2}\right)^2$ and $B = \pi r^2$.

The great mathematician Gauss observed that if one chooses the t_k 's carefully, one needs fewer of them. For example, suppose $n = 2$ in (3) and, to assume a kind of symmetry, suppose $\lambda_1 = \lambda_2 = \frac{1}{2}$. Then the remaining equations are

$$\frac{1}{2}(t_1 + t_2) = \frac{1}{2}, \quad \frac{1}{2}(t_1^2 + t_2^2) = \frac{1}{3}.$$

The first equation implies that $t_2 = 1 - t_1$ (nicely symmetric too), and the second one then becomes $t_1^2 + (1 - t_1)^2 = \frac{2}{3}$, or $2t_1^2 - 2t_1 + \frac{1}{3} = 0$. The quadratic formula gives two solutions for t_1 , and they add up to one, so one is t_1 and the other is t_2 :

$$t_1 = \frac{1}{2} - \frac{1}{\sqrt{12}} = \frac{3 - \sqrt{3}}{6} := \sigma, \quad t_2 = \frac{1}{2} + \frac{1}{\sqrt{12}} = \frac{3 + \sqrt{3}}{6} = 1 - \sigma.$$

This can be transformed into a similar volume formula, but the appearance of square roots makes it not so useful in practice.

But there are other solutions to (3) with $n = 2$! For example, suppose we decide we want to have $t_1 = 0$. Then the equations become

$$\lambda_1 + \lambda_2 = 1, \quad \lambda_2 t_2 = \frac{1}{2}, \quad \lambda_2 t_2^2 = \frac{1}{3}.$$

These are also easy to solve. Dividing the first by the second gives $t_2 = \frac{2}{3}$, and so $\lambda_2 = \frac{3}{4}$, and so $\lambda_1 = \frac{1}{4}$. Thus

$$\int_0^1 f(t) dt = \frac{1}{4} (f(0) + 3f(\frac{2}{3})).$$

We can apply symmetry here, even if it isn't apparent. Let $g(t) = f(1 - t)$. Then $\int_0^1 f(t) dt = \int_0^1 g(t) dt$ and we get

$$\int_0^1 f(t) dt = \frac{1}{4} (f(1) + 3f(\frac{1}{3}))$$

“for free”. We can also turn this into a volume rule for objects whose cross-sectional area is quadratic in their height: let N denote the area $2/3$ of the way up, then the volume is $\frac{H}{4}(B + 3N)$. The computations are somewhat trickier, and of course the formula is not symmetric.

Next step: what about stronger quadrature formulas? Do any of the formulas listed above have strength 3? That is to say, when is it true that for all $(\alpha, \beta, \gamma, \delta)$,

$$\int_0^1 (\alpha + \beta t + \gamma t^2 + \delta t^3) dt = \sum_{k=1}^n \lambda_k (\alpha + \beta t_k + \gamma t_k^2 + \delta t_k^3)?$$

If we've already verified strength 2, then all we have to check is that the coefficient of δ on both sides is equal:

$$\frac{1}{4} = \sum_{k=1}^n \lambda_k t_k^3.$$

There will be a mental drumroll as I look at the three examples given

$$\begin{aligned} \frac{1}{4} &= \frac{1}{6} \cdot 0 + \frac{4}{6} \cdot 18 + \frac{1}{6} \cdot 1, & (\text{Yes!}) \\ \frac{1}{4} &= \frac{1}{2} \cdot \left(\frac{1}{2} - \frac{1}{\sqrt{12}}\right)^3 + \frac{1}{2} \cdot \left(\frac{1}{2} + \frac{1}{\sqrt{12}}\right)^3, & (\text{Yes, believe it or not!}) \\ \frac{1}{4} &= \frac{1}{4} \cdot 0 + \frac{3}{4} \cdot \frac{8}{27}, & (\text{Alas, No!}) \end{aligned}$$

The two-point formula is remarkable: if f is a polynomial of degree ≤ 3 , then

$$\int_a^b f(x) dx = \frac{b-a}{2} (f(\sigma a + (1-\sigma)b) + f((1-\sigma)a + \sigma b)).$$

The general situation is this: if $0 \leq t_1 < \dots < t_{n+1} \leq 1$ are fixed, then there exist $\lambda_k \geq 0$ so that (1) is a quadrature formula of strength n . However, if you pick them *just* right, then you only need $\lfloor n/2 \rfloor + 1$ points. This sort of thing is studied in numerical analysis, a branch of applied mathematics.

I now offer you several “bonus” honor questions, due on the Monday after Thanksgiving, November 26.

1. Find **all** two-point quadrature formulas of strength 2. That is, parameterize the solutions to the system

$$\begin{aligned} 1 &= \lambda_1 + \lambda_2 \\ \frac{1}{2} &= \lambda_1 t_1 + \lambda_2 t_2 \\ \frac{1}{3} &= \lambda_1 t_1^2 + \lambda_2 t_2^2. \end{aligned}$$

Remark: it can be shown that any solution to these equations has $\lambda_1, \lambda_2 > 0$, so we can start by writing $\lambda_1 = \cos^2 \theta$ and $\lambda_2 = \sin^2 \theta$, which automatically handles the first equation and leaves two equations in three variables. In the end, you will want to express t_1 and t_2 in terms of θ . As a check, $\theta = \frac{\pi}{4}$ corresponds to $\lambda_1 = \lambda_2 = \frac{1}{2}$ and $\theta = \frac{\pi}{3}$ corresponds to $\lambda_1 = \frac{1}{4}$ and $\lambda_2 = \frac{3}{4}$.

2. Find a three-point quadrature formula of strength 4 for $[0,1]$. You will first have to determine the necessary equations, but they are similar to what we’ve already done. To simplify matters, you may look for a formula which is symmetric in $t \mapsto 1-t$; that is, you may assume that it is of the form

$$\int_0^1 f(t) dt = \lambda f\left(\frac{1}{2} - c\right) + \mu f\left(\frac{1}{2}\right) + \lambda f\left(\frac{1}{2} + c\right).$$

3. The idea of quadrature formulas generalizes to multiple integrals. (This is still an on-going subject of research, in fact.) Determine constants λ_k, μ so that

$$\int_0^1 \int_0^1 f(x, y) dA = \lambda_1 f(0, 0) + \lambda_2 f(0, 1) + \lambda_3 f(1, 0) + \lambda_4 f(1, 1) + \mu f\left(\frac{1}{2}, \frac{1}{2}\right)$$

for every polynomial $f(x, y) = \alpha + \beta x + \gamma y + \delta x^2 + \epsilon xy + \zeta y^2$.