Supplemental Quadrature Notes – Math 242, 11/16/01

The purpose of these notes is to illustrate the ideas with an additional example. We consider quadrature formulas of strength 1 on the interval $[0,1]$. This means a formula with the property that for all α, β, we have

$$\int_0^1 (\alpha + \beta x)\; dx = \sum_{k=1}^{n} \lambda_k(\alpha + \beta x_k)$$

$$\Longrightarrow\ \alpha + \tfrac{\beta}{2} = \alpha\left(\sum_{k=1}^{n}\lambda_k\right) + \beta\left(\sum_{k=1}^{n}\lambda_k x_k\right)$$

$$\Longrightarrow\ \sum_{k=1}^{n}\lambda_k = 1, \qquad \sum_{k=1}^{n}\lambda_k x_k = \tfrac{1}{2}.$$

Before I look at these equations, look at the trapezoid. The 'obvious" geometric formulas for the area give two immediate quadrature formulas of strength 1:

$$\int_0^1 f(x)\; dx = f\left(\tfrac{1}{2}\right) \qquad \int_0^1 f(x)\; dx = \tfrac{1}{2}f(0) + \tfrac{1}{2}f(1).$$

What would happen if we look at this systematically? If $n = 1$, then $\lambda_1 = 1, \lambda_1 x_1 = \frac{1}{2}$, and the first formula above just pops out. If $n = 2$, we get a system of two equations. Let's write $x_1 = r$ and $x_2 = s$, $r < s$. The equations are

$$\lambda_1 + \lambda_2 = 1, \qquad \lambda_1 r + \lambda_2 s = \tfrac{1}{2},$$

and these can be solved easily to give $\lambda_1 = \frac{s-\frac{1}{2}}{s-r}$ and $\lambda_2 = \frac{\frac{1}{2}-r}{s-r}$, and so there is a two-parameter family of quadrature formulas of strength 1:

$$\int_0^1 f(x)\; dx = \left(\frac{s-\frac{1}{2}}{s-r}\right) f(r) + \left(\frac{\frac{1}{2}-r}{s-r}\right) f(s).$$

Taking $r = 0$ and $s = 1$ gives the second formula. But there are lots of other examples. If we make (r,s) symmetric in $[0,1]$ by setting $s = 1-r$, then you can check that $\lambda_1 = \lambda_2 = \frac{1}{2}$. There are weirder solutions. If $r = -1$ and $s = 2$, then we still have $\lambda_1 = \lambda_2 = \frac{1}{2}$; taking $r = 4$ and $s = 5$ gives $\int_0^1 f(x)\; dx = \frac{9}{2}f(4) - \frac{7}{2}f(5)$. (Aesthetics favors $\lambda_k \geq 0$ whenever possible!)

Can any of these formulas have strength 2? Put $f(x) = x^2$ in there, to obtain

$$\tfrac{1}{3} = \frac{(s-\frac{1}{2})r^2 + (\frac{1}{2}-r)s^2}{s-r} = \frac{sr(r-s) + \frac{1}{2}(s+r)(s-r)}{s-r} = \tfrac{1}{2}(s+r) - rs$$

This becomes a quadratic in r,s: $6rs - 3r - 3s + 2 = 0$. One solution is $r = \frac{1}{2} - \frac{1}{\sqrt{12}}$ and $s = \frac{1}{2} + \frac{1}{\sqrt{12}}$. Another is to take $r = 0$, then $s = \frac{2}{3}$. In each case, the λ_k's are determined as above. Do these examples look familiar?