Solutions to Bonus Honor Questions – Math 242, 11/26/01

1. Find **all** two-point quadrature formulas of strength 2. That is, parameterize the solutions to the system

$$\begin{aligned} 1 &= \lambda_1 + \lambda_2 \\ \tfrac{1}{2} &= \lambda_1 t_1 + \lambda_2 t_2 \\ \tfrac{1}{3} &= \lambda_1 t_1^2 + \lambda_2 t_2^2. \end{aligned}$$

There are several possible approaches. The first one I gave away on the supplemental handout!. Writing $t_1 = r$ and $t_2 = s$, we solved the first two equations into

$$\lambda_1 = \frac{s - \frac{1}{2}}{s - r}, \qquad \lambda_2 = \frac{\frac{1}{2} - r}{s - r}.$$

I then asked what it takes for the the formula to have strength 2, which amounts to looking at the last equation. This becomes

$$6rs - 3r - 3s + 2 = 0 \implies s = \frac{3r - 2}{6r - 3}.$$

(If $r = \frac{1}{2}$, then this equation becomes $3s - \frac{3}{2} - 3s + 2 = 0$, which is impossible, so it's ok to divide by $6r - 3$.) Plugging for s into the formula above gives

$$\lambda_1 = \frac{1}{12r^2 - 12r + 4}, \qquad \lambda_2 = \frac{12r^2 - 12r + 3}{12r^2 - 12r + 4}.$$

In order to make this look prettier (not a *mathematical* requirement), we write $r = \frac{1}{2} + u$; then after some simplification,

$$s = \frac{1}{2} - \frac{1}{12u}, \qquad \lambda_1 = \frac{1}{1 + 12u^2}, \qquad \lambda_2 = \frac{12u^2}{1 + 12u^2}.$$

If you follow my hint, then $\lambda_1 = \cos^2\theta$ and $\lambda_2 = \sin^2\theta$ imply that $\tan^2\theta = 12u^2$, so $u = \frac{1}{\sqrt{12}}\tan\theta$ and $\frac{1}{12u} = \frac{1}{\sqrt{12}}\cot\theta$ and we have, ultimately, the quadrature formula:

$$\int_0^1 f(x)\, dx = (\cos^2\theta) f\left(\tfrac{1}{2} + \tfrac{1}{\sqrt{12}}\tan\theta\right) + (\sin^2\theta) f\left(\tfrac{1}{2} - \tfrac{1}{\sqrt{12}}\cot\theta\right).$$

Notice that the symmetric formula occurs when $\theta = \frac{\pi}{4}$. There are many different ways to express the final answer. I'll just mention one curious fact. If you multiply the first and third equations and subtract the square of the second, then

$$\begin{aligned} \frac{1}{3} - \frac{1}{4} &= (\lambda_1 + \lambda_2)(\lambda_1 t_1^2 + \lambda_2 t_2^2) - (\lambda_1 t_1 + \lambda_2 t_2)^2 \\ &= \lambda_1^2(t_1^2 - t_1^2) + \lambda_1\lambda_2(t_1^2 + t_2^2 - 2t_1t_2) + \lambda_2^2(t_2^2 - t_2^2) \\ \implies \frac{1}{12} &= \lambda_1\lambda_2(t_1 - t_2)^2. \end{aligned}$$

Using the previously solved λ_1 and λ_2 and noting that $t_1 - t_2 = s - r$, this last equation reduces to $-\frac{1}{12} = (r - \frac{1}{2})(s - \frac{1}{2})$, which turns into $6rs - 3r - 3s + 2 = 0$.

2. Find a three-point quadrature formula of strength 4 for [0,1]. To simplify matters, you may look for a formula which is symmetric in $t \mapsto 1 - t$; that is, you may assume that it is of the form

$$\int_0^1 f(t)\, dt = \lambda f(\tfrac{1}{2} - c) + \mu f(\tfrac{1}{2}) + \lambda f(\tfrac{1}{2} + c).$$

If the formula

$$\int_0^1 (\alpha + \beta t + \gamma t^2 + \delta t^3 + \epsilon t^4)\, dt = \sum_{k=1}^{n} \lambda_k (\alpha + \beta t_k + \gamma t_k^2 + \delta t_k^3 + \epsilon t_k^4)$$

holds for all $(\alpha, \beta, \gamma, \delta, \epsilon)$, then we have as before

$$\sum_{k=1}^{n} \lambda t_k^r = \tfrac{1}{r+1}, \qquad \text{for } r = 0, 1, 2, 3, 4.$$

With the substitutions suggested above, this becomes five equations:

$$\begin{aligned}
\lambda + \mu + \lambda &= 1\\
\lambda(\tfrac{1}{2} - c) + \mu \cdot \tfrac{1}{2} + \lambda(\tfrac{1}{2} + c) &= \tfrac{1}{2}\\
\lambda(\tfrac{1}{2} - c)^2 + \mu \cdot \tfrac{1}{4} + \lambda(\tfrac{1}{2} + c)^2 &= \tfrac{1}{3}\\
\lambda(\tfrac{1}{2} - c)^3 + \mu \cdot \tfrac{1}{8} + \lambda(\tfrac{1}{2} + c)^3 &= \tfrac{1}{4}\\
\lambda(\tfrac{1}{2} - c)^4 + \mu \cdot \tfrac{1}{16} + \lambda(\tfrac{1}{2} + c)^4 &= \tfrac{1}{5}.
\end{aligned}$$

A little algebraic simplification reduces these equations to:

$$\begin{aligned}
2\lambda + \mu &= 1\\
\lambda + \mu \cdot \tfrac{1}{2} &= \tfrac{1}{2}\\
\lambda(\tfrac{1}{2} + 2c^2) + \mu \cdot \tfrac{1}{4} &= \tfrac{1}{3}\\
\lambda(\tfrac{1}{4} + 3c^2) + \mu \cdot \tfrac{1}{8} &= \tfrac{1}{4}\\
\lambda(\tfrac{1}{8} + 3c^2 + 2c^4) + \mu \cdot \tfrac{1}{16} &= \tfrac{1}{5}
\end{aligned}$$

There are five equations in three variables, which should be disturbing; however, the second equation is a multiple of the first. From the first, we have $\mu = 1 - 2\lambda$, and after plugging this into the next three, we get

$$2\lambda c^2 = \frac{1}{12}, \qquad 3\lambda c^2 = \frac{1}{8}, \qquad 3\lambda c^2 + 2\lambda c^4 = \frac{11}{80}.$$

We're down to three equations in two unknowns, but now the first two are equivalent to $\lambda c^2 = \frac{1}{24}$, after which the third become $\lambda c^4 = \frac{1}{80}$. This implies that $c^2 = \frac{12}{80} = \frac{3}{20}$, so $\lambda = \frac{5}{18}$ and $\mu = \frac{4}{9}$. The formula, in all its glory, is

$$\int_0^1 f(t)\ dt = \tfrac{5}{18} f(\tfrac{1}{2} - \sqrt{\tfrac{3}{20}}) + \tfrac{4}{9} f(\tfrac{1}{2}) + \tfrac{5}{18} f(\tfrac{1}{2} + \sqrt{\tfrac{3}{20}}).$$

As a bonus, you can check that this formula actually has strength 5.

3. Determine constants λ_k, μ so that

$$\int_0^1 \int_0^1 f(x,y)\ dx\ dy = \lambda_1 f(0,0) + \lambda_2 f(0,1) + \lambda_3 f(1,0) + \lambda_4 f(1,1) + \mu f(\tfrac{1}{2}, \tfrac{1}{2})$$

for every polynomial $f(x,y) = \alpha + \beta x + \gamma y + \delta x^2 + \epsilon xy + \zeta y^2$.

Using the methods of Chapter 15, it's easy to see that

$$\int_0^1 \int_0^1 1\ dx\ dy = 1, \qquad \int_0^1 \int_0^1 x\ dx\ dy = \int_0^1 \int_0^1 y\ dx\ dy = \frac{1}{2},$$

$$\int_0^1 \int_0^1 x^2\ dx\ dy = \int_0^1 \int_0^1 y^2\ d\ xdy = \frac{1}{3}, \qquad \int_0^1 \int_0^1 xy\ dx\ dy = \frac{1}{4},$$

and so the desired equation is that

$$\begin{aligned} \alpha + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{3} + \frac{\epsilon}{4} + \frac{\zeta}{3} &= \lambda_1 \alpha + \lambda_2(\alpha + \gamma + \zeta) \\ &+ \lambda_3(\alpha + \beta + \delta) + \lambda_4(\alpha + \beta + \gamma + \delta + \epsilon + \zeta) \\ &+ \mu(\alpha + \tfrac{\beta}{2} + \tfrac{\gamma}{2} + \tfrac{\delta}{4} + \tfrac{\epsilon}{4} + \tfrac{\zeta}{4}). \end{aligned}$$

This looks terrible! But after considering the coefficients of the various greek letters in order, we find that this becomes

$$\begin{aligned} 1 &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \mu \\ \tfrac{1}{2} &= \lambda_3 + \lambda_4 + \tfrac{1}{2}\mu \\ \tfrac{1}{2} &= \lambda_2 + \lambda_4 + \tfrac{1}{2}\mu \\ \tfrac{1}{3} &= \lambda_3 + \lambda_4 + \tfrac{1}{4}\mu \\ \tfrac{1}{4} &= \lambda_4 + \tfrac{1}{4}\mu \\ \tfrac{1}{3} &= \lambda_2 + \lambda_4 + \tfrac{1}{4}\mu. \end{aligned}$$

These solve quickly. A comparison of the last three shows that $\lambda_2 = \lambda_3 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$, hence $\lambda_4 + \frac{1}{2}\mu = \frac{1}{2} - \frac{1}{12} = \frac{5}{12}$, and so $(\frac{1}{2} - \frac{1}{4})\mu = \frac{5}{12} - \frac{1}{4} = \frac{1}{6}$, so that $\mu = \frac{2}{3}$ and $\lambda_4 = \frac{1}{12}$. The first equation now implies that $\lambda_1 = \frac{1}{12}$, so symmetry abounds. You might want to check if this formula has strength 3.

$$\int_0^1 \int_0^1 f(x,y)\ dx\ dy = \frac{f(0,0) + f(0,1) + f(1,0) + f(1,1) + 8f(\frac{1}{2}, \frac{1}{2})}{12}.$$