Math 496 **Notes on a greedy example** **Spring 2006**

Suppose x is a positive rational number and N is a positive integer. I want to show that there exists a finite sequence of integers

$$N \leq N_1 < N_2 < \cdots < N_r$$

such that

$$\frac{1}{N_1} + \frac{1}{N_2} + \cdots + \frac{1}{N_r} = x.$$

The proof comes in two parts. In the first, we use the divergence of the harmonic series to assert that there exists M so that

$$\frac{1}{N} + \frac{1}{N+1} + \cdots + \frac{1}{M} < x \leq \frac{1}{N} + \frac{1}{N+1} + \cdots + \frac{1}{M} + \frac{1}{M+1}.$$

Now, let us write

$$x - \left(\frac{1}{N} + \frac{1}{N+1} + \cdots + \frac{1}{M}\right) = \frac{a}{b} > 0.$$

By hypothesis,

$$\frac{a}{b} \leq \frac{1}{M+1}$$

We have $b \geq (M+1)a$, so by the division algorithm, we can write $b = (M+k)a + r$, with $k \geq 1$ and $0 \leq a \leq r - 1$. If $r = 0$, then

$$\frac{a}{b} = \frac{a}{a(M+k)} = \frac{1}{M+k},$$

and the algorithm is complete. If $r > 0$, then

$$\frac{a}{b} = \frac{a}{a(M+k)+r} = \frac{1}{M+k+1} + \frac{(a-r)}{(a(M+k)+r)(M+k+1)}.$$

That is,

$$x = \frac{1}{N} + \frac{1}{N+1} + \cdots + \frac{1}{M} + \frac{1}{M+k+1} + \frac{a'}{b'},$$

where

$$\frac{a'}{b'} = \frac{(a-r)}{(a(M+k)+r)(M+k+1)}.$$

Since $a' < a$. We can repeat this argument, and the algorithm will terminate in at most a steps.