Math 230 **Honors Question 5** **Due Fri., Oct. 21, 2005**

This is a series of questions related to the numbers

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}; \qquad H_n = \ln n + \gamma_n,$$

where $\lim_{n\to\infty} \gamma_n = \gamma$.

a. Find an expression for

$$E_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}$$

in the form $E_n = \alpha H_m$ for an appropriate constant α and integer m, which depends on n. Hint: this is extremely easy!

b. Find an expression for

$$O_n = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2n+1}$$

in the form $O_n = \beta_1 H_{m_1} + \beta_2 H_{m_2}$ for appropriate constant β_1, β_2 and integers m_1, m_2, which depend on n. Hint: look at $E_n + O_n$.

c. Let $a_n = (-1)^{n+1}\frac{1}{n}$ and let

$$S_N = \sum_{n=1}^{N} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \cdots + (-1)^{N+1}\frac{1}{N}.$$

Find an exact formula for S_N in terms of various H_m's. You should have a slightly different answer for $N = 2r$ and $N = 2r + 1$.

d. Use your answer in c. to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

is a convergent series whose sum is $\ln 2$. (This will require writing $H_n = \ln n + \gamma_n$.)

e. Use these techniques to explore the convergence (or not) of

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} \cdots$$

in which two positive fractions are followed by a negative fraction. The exact formula for the partial sums will be slightly different for $N = 3r,\ 3r + 1$ and $3r + 2$.

f. Same question for

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} \cdots.$$

g (Extra Credit). Same question for

$$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{6} + \cdots$$

where there is always 1 negative term and 2^k positive terms for $k = 0, 1, 2, \ldots$. This one is more complicated. Hint: it's enough to compute the partial sum after the k-th batch of positive terms.