domain containing $|z| \le 1$, Then

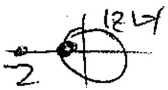
$$\max_{|z|\le 1} |f(z)| = \max_{|z|=1} |f(z)|.$$

This is true for $f(z) = \frac{z^2}{z+2}$

and $|z|=1 \Rightarrow$

$$|f(z)| = \frac{|z^2|}{|z+2|} = \frac{1}{|z+2|}$$

The maximum of $|f(z)|$ occurs when $|z+2| = |z-(-2)|$ is minimized, and this occurs at $z=-1$



$$|f(z)| \le \frac{1}{|-1+2|} = 1.$$

---

§3.3-1 This is awful to do directly, though if $z=z_1$ or z_3, cancellation in the numerator + denominator give $L(z_1)=w_1$, $L(z_3)=w_3$ easily.

By the cross-ratio Theorem done in class (§3.3-16) $L(z)$ is defined explicitly by

$$\frac{z-z_1}{z-z_3}\cdot\frac{z_2-z_1}{z_2-z_3} = \frac{L(z)-w_1}{L(z)-w_3}\cdot\frac{w_2-w_1}{w_2-w_3}$$

(to get this to work, I'm permuting (z_2,z_3) and (w_2,w_3) in #16).

i.e. $\frac{z-z_1}{z-z_3}\cdot\alpha = \frac{L(z)-w_1}{L(z)-w_3}\cdot\beta$

$\Rightarrow \alpha(z-z_1)(L(z)-w_3) = \beta(z-z_3)(L(z)-w_1)$

Solving for $L(z)$ gives the formula. I'm glad I didn't have to grade this

---

§3.3-1a. $T(0)=0$, $T(1)=1$, $T(i)=\infty$ $\quad T(z) = \frac{az+b}{cz+d}$

Many ways to do it. From the 1st and 3rd facts, $T(z) = \alpha\cdot\frac{z}{z-i}$

then $1 = T(1) = \alpha\cdot\frac{1}{1-i} \Rightarrow \alpha = 1-i$

$$T(z) = \frac{(1-i)z}{z-i}$$

Math 440 HW 11 Due 11/26/07

(a). $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$ for all z, so

$$e^z - 1 = \sum_{n=1}^{\infty}\frac{z^n}{n!} = z + \frac{z^2}{2!} + \frac{z^3}{3!}$$

and $|e^z-1| = |z|\,\left|1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots\right|$

As noted in class, if $|z|\le 1$, then

$$\left|1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots\right| \ge 1 - \frac{|z|}{2!} - \frac{|z|^2}{3!} - \frac{|z|^3}{4!}$$

and $(k+1)! = (k+1)\cdot k \cdots 2\cdot 1 \ge 2^k$.

Putting this all together, if $|z| = r < 1$ then

$$|e^z-1| \ge r\cdot\left(1 - \frac{r}{2!} - \frac{r^2}{3!} - \frac{r^3}{4!}\cdots\right)$$

$$\ge r\left(1 - \frac{r}{2} - \frac{r^2}{2^2} - \frac{r^3}{2^3} - \cdots\right)$$

$$= r\cdot\left(1 - \frac{r/2}{1-r/2}\right)$$

$$= r\left(1 - \frac{r}{2-r}\right) = r - \frac{r^2}{2-r}$$

[Note: $\frac{-r^k}{(k+1)!} \ge \frac{-r^k}{2^k}$ since $(k+1)! \ge 2^n$]

(b) Suppose now that w is given and $|w| < \frac{1}{6}$

Consider the function $f(z) = z(e^z-1)$ on the circle $|z| = \frac{1}{2}$.

$|f(z)| = |z|\cdot|e^z-1| = \frac{1}{2}\cdot|e^z-1|$

By 2a, $|e^z-1| \ge \frac{1}{2} - \frac{1/4}{2-1/2} = \frac{1}{2} - \frac{\frac{1}{4}}{\frac{3}{2}} = \frac{1}{2} - \frac{1}{6}$

$= \frac{1}{3}$ on $|z| = \frac{1}{2}$

so $|f(z)| \ge \frac{1}{6} > |w|$ on $|z| = \frac{1}{2}$

This means, by Rouché's Theorem, that $f(z)$ and $f(z)-w$ have the same number of solutions inside $|z|=\frac{1}{2}$. But $f(z) = z(e^z-1) = 0$ $\Rightarrow z=0$ or $e^z-1=0$, $z = n\cdot 2\pi i$, only $n=0$ is inside $|z|=1/2$, so this is a zero of multiplicity two, so $f(z)-w$ also has two solutions, counting multiplicity as always.

5.3 14

#3 $T(i) = i$, $T(-1) = 1$ $\qquad T(z) = \frac{az+b}{cz+d}$

$T(i)$: $\frac{a+b}{c+d} = -1 \qquad \frac{-a+b}{-c+d} = 1 \leftarrow T(-1)$

$$\Rightarrow \left.\begin{array}{l} a+b+c+d=0 \\ -a+b+c-d=0 \end{array}\right\} \Rightarrow \begin{array}{l} c=-b \\ d=-a \end{array}$$

$$i = T(i) = \frac{a\cdot i + b}{-b\cdot i - a}$$

$$b - ai = ai + b \Rightarrow 2ai = 0$$

$\Rightarrow a = 0$, so $d = 0$ and

$$T(z) = \frac{b}{-bz} = -\frac{1}{z}.$$

4c. $T(1) = 1$
$T(0) = 0$
$T(i) = 1+i$

$$T(z) = \frac{az+b}{cz+d}$$

$$1 = \frac{a+b}{c+d} \qquad 0 = \frac{b}{d}$$

$$\Rightarrow b = 0, \quad a = c+d$$

$$T(z) = \frac{(c+d)z}{cz+d}$$

so $1+i = \frac{(c+d)i}{ci+d}$

$$c(-1+i) + d(1+i) = ci + di$$

$$c(-1) + d = 0 \Rightarrow c = d.$$

$$T(z) = \frac{2z}{z+1}$$

(Check: $T(i) = \frac{2i}{1+i}\cdot\frac{1-i}{1-i} = \frac{2+2i}{2} = 1+i$ ✓.

There are many correct ways to do these.

In fact

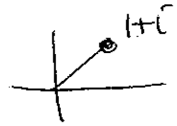
$T(\cos t + i\sin t)$
$= \frac{1-\cos t - \sin t}{1-\sin t}(1-i)$

$f(z) = z^i = e^{i\,\mathrm{Log}\,z}$ at $z = 1+i$

$$f(z) = f(1+i) + f'(1+i)(z-(1+i)) + \frac{f''(1+i)}{2!}(z-(1+i))^2 + \cdots$$

so we need find the indicated values

$$f(1+i) = e^{i\,\mathrm{Log}(1+i)}$$



$$\mathrm{Log}(1+i) = \ln\sqrt{2} + i\cdot\frac{\pi}{4}$$

$$\text{so } f(1+i) = e^{i(\ln\sqrt{2} + i\frac{\pi}{4})} = e^{-\frac{\pi}{4} + i\ln\sqrt{2}}$$

$$f'(z) = e^{i\,\mathrm{Log}\,z}\cdot(i\,\mathrm{Log}\,z)' = e^{i\,\mathrm{Log}\,z}\cdot\frac{i}{z}$$

$$\text{so } f'(1+i) = \frac{i}{1+i}\cdot e^{-\frac{\pi}{4}+i\ln\sqrt{2}}$$

$$f''(z) = e^{i\,\mathrm{Log}\,z}\left(\frac{i}{z}\right)^2 + e^{i\,\mathrm{Log}}\left(\frac{-i}{z^2}\right) = e^{i\,\mathrm{Log}\,z}\cdot\frac{-1-i}{z^2}, \text{ so}$$

$$f''(1+i) = -\frac{1+i}{(1+i)^2}\cdot e^{-\frac{\pi}{4}+i\ln\sqrt{2}}$$

You can simplify if you want.

#5 $T(z) = \frac{az+b}{cz+d}$

one of many ways.

$T(1) = 0 \Rightarrow a+b = 0$
$T(i) = \infty \Rightarrow ci+d = 0$
$T(\infty) = 2 \Rightarrow \frac{a}{c} = 2$

so $b = -a$, $c = \frac{a}{2}$, $d = -\frac{ai}{2}$

$$T(z) = \frac{az - a}{\frac{a}{2}z - \frac{ai}{2}} = \frac{2(z-1)}{(z-i)}$$

$T(1) = 0$, $T(i) = \infty$,

$$T(-1) = \frac{2(-1-1)}{-1-i} = \frac{4}{1+i}\cdot\frac{1-i}{1-i} = \frac{4-4i}{2} = 2-2i$$

so image of $x^2+y^2=1$ is circle/line containing $0, \infty, 2-2i$. This is the line $u+v=0$.

$T(z) = \frac{i}{z+1}$, $c > 0$

What is the image of the line $x = c$?

$T(c) = \frac{i}{1+c}$ $\quad T(\infty) = 0$

(note ∞ is on every line)

if $z = c + iy$

$$T(z) = \frac{i}{1+c+iy} \cdot \frac{1+c-iy}{1+c-iy}$$

$$= \frac{y + i(1+c)}{(1+c)^2 + y^2}$$

$$u = \frac{y}{(1+c)^2+y^2} \quad v = \frac{1+c}{(1+c)^2+y^2}$$

$$\text{so } u^2+v^2 = \frac{y^2+(1+c)^2}{((1+c)^2+y^2)^2} = \frac{1}{(1+c)^2+y^2}$$

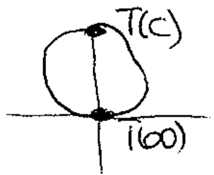
That is, $(u^2+v^2)(1+c) = v$

$$u^2+v^2 - \frac{1}{1+c}v = 0$$

$$u^2 + \left(v - \frac{1}{2(1+c)}\right)^2 = \frac{1}{2(1+c)^2}$$

This is the circle with

center $\left(0, \frac{1}{2(1+c)}\right)$ and radius $\frac{1}{2(1+c)}$



b. $T(0) = \frac{i}{1+0} = i$

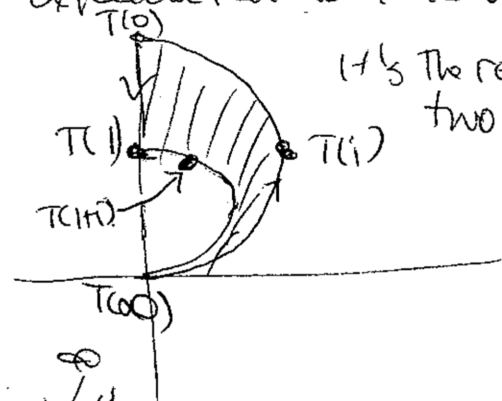
$T(1) = \frac{i}{1+1} = \frac{i}{2}$

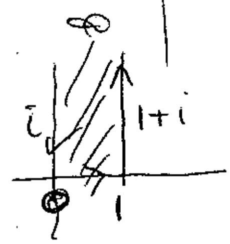
$T(i) = \frac{i}{1+i} \cdot \frac{1-i}{1-i} = \frac{1+i}{2}$

$T(1+i) = \frac{i}{2+i} \cdot \frac{2-i}{2-i} = \frac{1+2i}{5}$

$T(\infty) = \frac{i}{\infty+i} = 0$

The answer to b) is below, with explanation to follow.





if x is real, then $T(x) = \frac{i}{x+1}$ is on the imaginary axis, as $0 \le x \le 1$, you go down from i to $\frac{i}{2}$.

The image of $x=1$ is the circle with center $(0, \frac{1}{4})$ and radius $\frac{1}{4}$ from $c=1$ as indicated. Following the line takes you to $T(\infty) = 0$.

If z goes along the imaginary axis $x = 0$ is the circle with center $(0, \frac{1}{2})$ and radius $\frac{1}{2}$ from the first part.

Finally, how do you know you get the inside and not the outside?

Many ways: easiest. $T(-1) = \infty$

-1 is outside the half strip, so $T(-1)$ has to be outside the image.

On $|z|=1$, $|g(z)| = |2+i||z|^2 = \sqrt{5}\cdot 1$

$1 \le |f(z)| \le 2$,

so $|g(z)| > |f(z)|$ on $|z|=1$

Hence $g(z) - f(z) = (2+i)z^2 - f(z)$ has exactly as many zeros counting multiplicity as $g(z)$. But g has a double zero at $z=0$, hence $h(z) = (2+i)z^2 - f(z)$ has two zeros inside $|z|=1$

We are told that $h(z)$ has exactly one zero in $\mathbb{C}$ at $z=z_0$, so we must have

$h'(z_0) = 0$!

$h'(z) = 2(2+i)z - f'(z)$, so

$0 = (4+2i)z_0 - f'(z_0)$ and

$\underline{f'(z_0) = (4+2i)z_0}$

---

9. Let $f(z) = (z-1)(z+1)g(z)$

Then g has removable singularities at $z = \pm 1$ because $f(1)=0$, $f(-1)=0$

and $4 = f(0) = -g(0)$. It follows that there exists z_0, $|z_0|=2$ with $|g(z_0)| \ge 4$. But then

$|f(z_0)| = |z^2-1|\cdot|g(z_0)| \ge (2^2-1)4 = 12$

If $f(2) = 12i$, then $g(2) = 4i$ and you can't say anything else, as per my email.

---

10. By the basic rule of mystery novels and movies, $T(r_4)$ has to be r_3. What else could it be?

As for a proof...

Proof 1

$$\frac{z-r_1}{z-r_2}\cdot\frac{r_3-r_2}{r_3-r_1} = \frac{T(z)-T(r_1)}{T(z)-T(r_2)}\cdot\frac{T(r_3)-T(r_2)}{T(r_3)-T(r_1)}$$

for all z $$= \frac{T(z)-r_2}{T(z)-r_1}\cdot\frac{r_4-r_1}{r_4-r_2}$$

If $z = r_4$, then

$$\cancel{\frac{r_4-r_1}{r_4-r_2}}\cdot\frac{r_3-r_2}{r_3-r_1} = \frac{T(r_4)-r_2}{T(r_4)-r_1}\cdot\cancel{\frac{r_4-r_1}{r_4-r_2}}$$

$$(r_3-r_2)T(r_4) - r_1(r_3-r_2) = (r_3-r_1)T(r_4) - r_2(r_3-r_1)$$

$$(r_1-r_2)T(r_4) = r_1r_3 - \cancel{r_1r_2} - r_2r_3 + \cancel{r_1r_2} = (r_1-r_2)r_3$$

so $T(r_4) = r_3$.

Proof 2

For every T there exists z_0 so that $T(z_0) = z_0$ and, evidently, $z_0 \notin \{r$

Let $U(z) = T(T(z))$.

Then $U(z_0) = T(T(z_0)) = T(z_0) = z_0$

$U(r_1) = T(T(r_1)) = T(r_2) = r_1$

$U(r_2) = T(T(r_2)) = T(r_1) = r_2$

so U has three different fixed points and as shown in class, this implies that $U(z) = z$ for all z.

so $U(r_3) = T(T(r_3)) = T(r_4) = r_3$.

#1,2.

It turns out that on $|z|=\frac{1}{2}$, $|e^z-1|$ is minimized when $z=-\frac{1}{2}$, and $|e^{-\frac{1}{2}}-1|=.39346\ldots$ which is larger than $\frac{1}{3}$, so Rouché's Theorem says that $z(e^z-1)=w$ has two solutions when $|w|<\frac{1-\frac{1}{\sqrt{e}}}{2}\simeq .1967\ldots$ (cont'd below). which is larger than $\frac{1}{6}$.

#3 Everyone got it. You "should" be able to do this without mechanical help.

#4. A mess. We didn't actually prove that $\frac{d}{dz}(z^\alpha)=\alpha z^{\alpha-1}$ for $\alpha\in\mathbb{C}$ (at least I don't remember it)

To do this carefully.

$z^\alpha=e^{\alpha\log z}$ for some log. branch.

$$\frac{d}{dz}(z^\alpha)=e^{\alpha\log z}\cdot(\alpha\log z)'$$
$$=e^{\alpha\log z}\cdot\frac{\alpha}{z}$$
$$=\alpha\cdot e^{\alpha\log z}e^{-1\cdot\log z}$$
$$=\alpha\cdot e^{(\alpha-1)\log z}$$

#5 See #3.

#1,2 What is the philosophical role of a "hint"? Can you simply accept assertions in a hint without explaining why they're true. I appear to be a minority here in thinking "no". Oh well.

#6,7. This was a hard final problem - you did well on this.

Math 448
HW 11
More

#8 This was a really hard final problem. I'll do it in class

#9 A botched problem, but

If $f(z)=(z^2-1)g(z)$

Then $|f(z)|=|z^2-1|\cdot|g(z)|$

On $|z|=2$, $|z^2-1|\geq|z|^2-1=2^2-1=3$

To elaborate on my error, if
$f(1)=0$, $f(0)=4$, $f(-1)=0$, $f(2)=2i$

and $h(z)=f(z)+\lambda(z-1)\cdot z(z+1)(z-2)$

Then h will have the same values as above, and $h(2i)=f(2i)+\lambda\cdot(2i-1)\cdot 2i\cdot(2i+1)(2i-2)$ can have any value you like.

#10 One more solution.

Let $T(z)=\frac{az+b}{cz+d}$

Then $T(r_1)=r_2$ and $T(r_2)=r_1\Rightarrow$

$r_2(cr_1+d)=ar_1+b$
$r_1(cr_2+d)=ar_2+b$

subtracting

$$d(r_2-r_1)=a(r_1-r_2)$$
$$\Rightarrow d=-a$$

$\Rightarrow T(z)=\frac{az+b}{cz-a}$. From here, $T(T(z))=z$.

Using the matrix representation.

$$\begin{bmatrix}a&b\\c&-a\end{bmatrix}\begin{bmatrix}a&b\\c&-a\end{bmatrix}=\begin{bmatrix}a^2+bc&ab-ab\\ac-ac&bc+a^2\end{bmatrix}$$
$$=\begin{bmatrix}a^2+bc&0\\0&a^2+bc\end{bmatrix}\text{ so } T(T(z))=\frac{(a^2+bc)z}{(a^2+bc)}=z$$

and $T(r_3)=r_4\Rightarrow T(r_4)=r_3$