The ideas of calculus – no formulas, just words.
Math 221E Fall 2009 – Prof. Bruce Reznick

1. A problem in calculus is regarded as solved it is reduced to a problem in high school algebra. A calculus course devotes much time to practicing the necessary algebraic computations and to developing a repertoire of functions (polynomial, trigonometric, exponential, logarithmic and combinations thereof) to which these computations can be applied. Students often get the misleading impression that these activities are what calculus is all about. You know you understand calculus when you can compute as a means to an end, much as you drive a car or ride a bicycle.

2. Geometry is meaningful. Cartesian coordinates give the label of a pair of numbers to each point of the plane. The set of points whose coordinates bear a simple relationship to each other describes a simple curve, and vice versa. Information about the geometric relationship of the points on the graph of a functions usually gives important information about the function itself. This is rather amazing if you think about it. These relationships also apply to functions of several variables, and their graphs, which are surfaces and more complicated geometric objects.

3. The derivative represents both the rate of change of a function (if it's defined) and the slope of the tangent line of its graph (if it's defined). This equivalence illustrates the last paragraph. The integral represents both the cumulative effect of a function and the area under its graph. This equivalence also illustrates the last paragraph.

4. In the most fundamental way, the derivative and the integral are inverses of each other, and information about one is telling about the other. Their technical definitions are quite subtle and involve limits. Most of the material taught in a calculus course was developed before limits, although the original explanations are no longer considered satisfactory.

5. The derivative obeys a small set of rules (linearity, chain, product, etc.) and large numbers of derivatives can be computed once you know a few basic ones. The rules for the derivative imply rules for the integral (linearity, substitution, integration by parts), so that, less easily, a somewhat smaller number of integrals can be computed. Differentiation requires skill; integration requires art; understanding limits requires patience and experience.

6. The mathematical problems for which calculus can be used include, but are not limited to, the geometric (computing areas, volume, surface area, arc length, etc.) and the extremal (finding maximal, minimal and otherwise optimal solutions). A reasonable function reaches its extremes where its derivative is zero, where its derivative does not exist, or on its boundary.

7. A mathematical model is a metaphor for an aspect of reality, in which certain measurements (position, velocity, profit, temperature, size, etc.) testify for the whole. Calculus is applied to the functions appearing in mathematical models, not to reality itself. Calculus leads to predictions within the model about the associated measurements, and thereby to predictions about reality. A model is considered good if these predictions are accurate. Constructing good models is very difficult, and usually not the job of the mathematician, at least this one.