

Almost Alternating Sums

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August 5, 2003

The behavior of the sum

$$S_N(\alpha) := \sum_{n=1}^N (-1)^{\lfloor n\alpha \rfloor}$$

as $N \rightarrow \infty$ is not transparent. The random walk $\sum_{n=1}^N w_n$, where the w_n are independent random variables taking the values ± 1 with equal probability, is known [21] to typically have absolute value around $c\sqrt{N}$, for an appropriate constant c and large N . Knowing this, and knowing that for irrational α the sequence $\lfloor n\alpha \rfloor$ is “random-ish” modulo 2, a natural guess is that $|S_N(\alpha)|$ is also around \sqrt{N} .

Contrary to this expectation, for almost all real numbers α

$$|S_N(\alpha)| \leq (\log N)^2 \tag{1}$$

for all large N . This is a corollary of a theorem of Khintchine, which we state precisely in Section 1.

We devote the bulk of this article to two elementary proofs that

$$|S_N(\alpha)| \leq \frac{\log N}{2 \log(1 + \sqrt{2})} + 1 \tag{2}$$

for all N and an explicit countable set of α , including $\sqrt{2}$ and $\sqrt{5} + 1$. Our first proof is entirely self-contained and is given in Section 2. In Section 3, we give a second proof which, while elementary, makes use of continued fractions. This proof applies to an uncountable set of α including, for example, $\frac{2}{e-1}$. Moreover, it can be adapted to show that there are infinitely many N such that

$$|S_N(\sqrt{2})| > \frac{\log N}{2 \log(1 + \sqrt{2})} + 0.78.$$

This means that as $N \rightarrow \infty$, the first constant in (2) is sharp (at least for $\alpha = \sqrt{2}$) and the second cannot be improved even to $3/4$.

The situation for rational α is more clear. As shown by the third author [20], for rational α the limit

$$\lim_{N \rightarrow \infty} \frac{S_N(\alpha)}{N}$$

is well-defined. Moreover, if this limit is zero then $S_N(\alpha)$ is bounded and periodic.

In Figure 1, we show the points $(N, S_N(\sqrt{2}))$ for $0 \leq N \leq 238$. It is already apparent in this figure that $S_N(\sqrt{2})$ is not behaving like a random walk: there are never three consecutive “up” steps. Also, the graph is symmetric around the peak at $N = 119$, i.e., $S_N(\sqrt{2}) = S_{238-N}(\sqrt{2})$ for those N pictured in Figure 1.

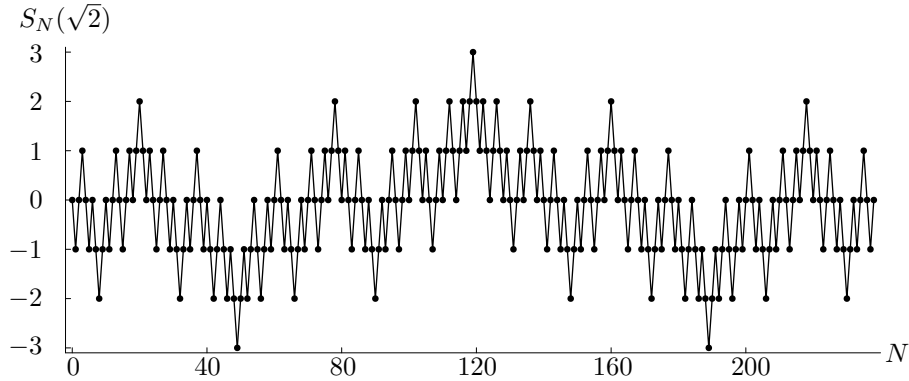


Figure 1: The points $(N, S_N(\sqrt{2}))$ have been connected for visual clarity.

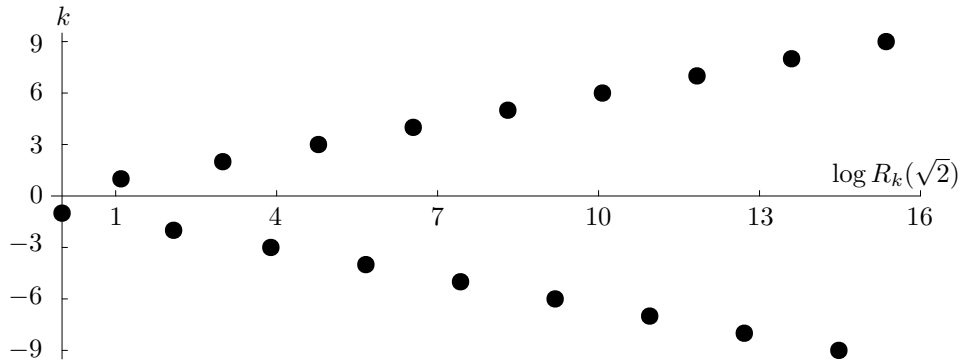


Figure 2: The points $(\log R_k(\sqrt{2}), k)$, for $-9 \leq k \leq 9$, except $k = 0$.

If we restrict our attention to just the *record-holders* — those N for which $S_N(\sqrt{2})$ takes on a value for the first time — another aspect of the structure of $S_N(\sqrt{2})$ becomes apparent. For the sake of rigor, we define the record-holder at the integer k by

$$R_k(\alpha) := \min\{N \geq 0: S_N(\alpha) = k\}.$$

In Figure 2, we plot the points $(\log R_k(\sqrt{2}), k)$ for $-9 \leq k \leq 9$ (except $k = 0$). The points approach two lines; we show in Corollary 3.3 that $R_k(\sqrt{2})$ is $\lfloor \frac{1}{4}(\sqrt{2} + 1)^{2k+1} \rfloor$ if k is positive, and $\lfloor \frac{1}{4}(\sqrt{2} + 1)^{-2k} \rfloor$ if k is negative.

Our proofs do not give a logarithmic bound on $|S_N(\alpha)|$ for general α ; indeed, for $\alpha = \pi \approx 3.14159$ we do not believe that a logarithmic bound is true. For $N \leq 10^7$, computations reveal that $-22 \leq S_N(\pi) \leq 3$. We are unaware of any non-trivial bound on $|S_N(\pi)|$. The record holders $R_k(\pi)$ are plotted in Figure 3. We note that the asymmetry and irregular clumping of points in Figure 3 seems to be more typical than the orderliness depicted in Figure 2.

In Section 1, we prove that $|S_N(\alpha)| \leq C_\alpha \log N$ (for quadratic irrational α) using the theory of discrepancy, which we define but do not pursue further. Section 1 is provided for historical background and as a hook into related literature; the remainder of this article is logically independent of Section 1. In Section 2, we give

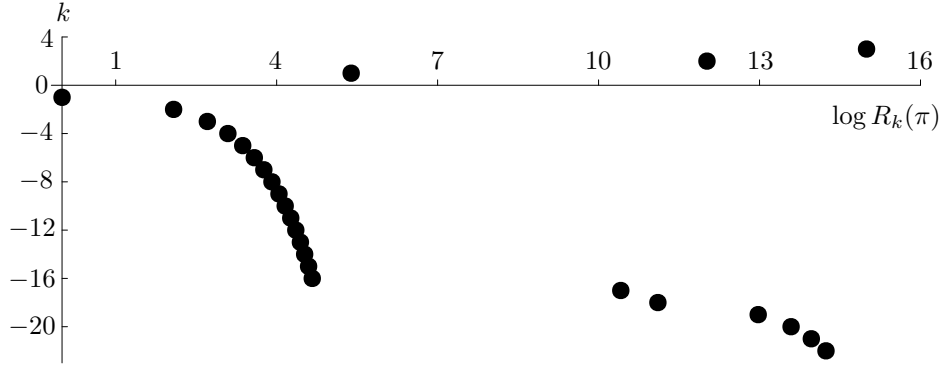


Figure 3: The points $(\log R_k(\pi), k)$, for $-22 \leq k \leq 3$, except $k = 0$.

an elementary bound on $|S_N(\alpha)|$ for a countable set of α . In Section 3, we introduce the required facts and definitions about continued fractions, and give a formula for $S_N(\alpha)$ for an uncountable set of α . From this formula, we get upper and lower bounds on the growth of $S_N(\alpha)$, and prove the formulas for $R_k(\sqrt{2})$ given above. We conclude in Section 4 with some questions we have not been able to answer.

Before we begin the analysis, we introduce some notation. The natural numbers begin at 1, i.e., $\mathbb{N} = \{1, 2, 3, \dots\}$. The fractional part of x is given by $\{x\} := x - \lfloor x \rfloor$. We make use of big-Oh notation only with respect to N ; that is, $f(N) = \mathcal{O}(g(N))$ if there is a constant C such that $|f(N)| \leq Cg(N)$ for all sufficiently large N . Note that C may depend on α , but does not depend on N . We also make lavish use of Iverson's notation

$$\llbracket Q \rrbracket := \begin{cases} 1, & \text{the statement } Q \text{ is true;} \\ 0, & \text{the statement } Q \text{ is false.} \end{cases}$$

See Knuth [14] for an eloquent, award-winning argument in favor of this notation¹. Iverson used (Q) , and Knuth uses $[Q]$, but we find $\llbracket Q \rrbracket$ visually appealing, especially in light of the other frequent uses for parentheses and brackets.

1 Discrepancy

An enduring topic in number theory has been the discrepancy shown between the expected and actual behavior of a sequence in the interval $[0, 1)$. For example, if $w_n \in [0, 1)$ for $n \in \mathbb{N}$ is uniformly random, one would expect that it is in the interval $[0, x)$ with frequency x . The difference between the actual and expected behavior is formally measured by

$$D_N^*(a_n, x) := \left| x - \frac{1}{N} \sum_{n=1}^N \llbracket w_n < x \rrbracket \right|,$$

and the discrepancy of the sequence is denoted

$$D_N^*(a_n) := \sup_{0 \leq x < 1} D_N^*(w_n, x).$$

¹See <http://www.maa.org/awards/ford.html> for a description of the Lester R. Ford Award for mathematical exposition, and the list of award-winning articles.

Discrepancy plays no role in the proofs of (2) given in Sections 2 and 3. However, the powerful results (stated below) of Behnke, Schmidt, and Khintchine regarding the discrepancy of $(\{n\alpha\})$ imply bounds on $S_N(\alpha)$ which are more general but less precise than those given elsewhere in this article.

We now show the connection between $S_N(\alpha)$ and the discrepancy of fractional part sequences. First, observe that $\lfloor n\alpha \rfloor$ is even exactly if there is an integer k with $2k \leq n\alpha < 2k + 1$, which is the same as $k \leq n\frac{\alpha}{2} < k + \frac{1}{2}$. In other words,

$$\llbracket \lfloor n\alpha \rfloor \text{ is even} \rrbracket = \llbracket k \leq n\frac{\alpha}{2} < k + \frac{1}{2} \rrbracket = \llbracket \{n\frac{\alpha}{2}\} < \frac{1}{2} \rrbracket$$

With one eye on the definition of $D_N^*(\alpha)$, we now rewrite the definition of $S_N(\alpha)$ using $(-1)^{\lfloor n\alpha \rfloor} = \llbracket \lfloor n\alpha \rfloor \text{ is even} \rrbracket - \llbracket \lfloor n\alpha \rfloor \text{ is odd} \rrbracket$:

$$S_N(\alpha) = \sum_{n=1}^N \llbracket \lfloor n\alpha \rfloor \text{ is even} \rrbracket - \llbracket \lfloor n\alpha \rfloor \text{ is odd} \rrbracket.$$

Add this equation to the obvious

$$N = \sum_{n=1}^N \llbracket \lfloor n\alpha \rfloor \text{ is even} \rrbracket + \llbracket \lfloor n\alpha \rfloor \text{ is odd} \rrbracket$$

to arrive at

$$S_N(\alpha) = 2 \left(\sum_{n=1}^N \llbracket \lfloor n\alpha \rfloor \text{ is even} \rrbracket \right) - N = -2N \left(\frac{1}{2} - \frac{1}{N} \sum_{n=1}^N \llbracket \{n\frac{\alpha}{2}\} < \frac{1}{2} \rrbracket \right).$$

Thus,

$$|S_N(\alpha)| = 2N D_N^* \left(\{n\frac{\alpha}{2}\}, \frac{1}{2} \right) \leq 2N D_N^* (\{n\frac{\alpha}{2}\}). \quad (3)$$

There are several accounts of the theory underlying discrepancy, most notably the colorful introductory book of Hlawka [11] and the recent — and encyclopedic — treatise of Drmota & Tichy [6]. There are three results that are of particular interest here.

- Behnke [2, 6, Cor. 1.65; 19, 3] classified those α for which

$$D_N^* (\{n\alpha\}) = \mathcal{O} \left(\frac{\log N}{N} \right).$$

In particular, this is true of all quadratic irrationals and is *not* true of $\frac{1}{e-1}$.

- Schmidt [6, Thm. 1.51; 18] showed that for any α , there are infinitely many N with

$$D_N^* (\{n\alpha\}) > \frac{1}{100} \frac{\log N}{N}.$$

- Let $\psi(n)$ be any positive increasing function. Khintchine [6, Thm. 1.72; 12, 13] proved that

$$D_N^* (\{n\alpha\}) = \mathcal{O} \left(\frac{\log N}{N} \psi(\log N) \right)$$

for almost all real numbers α if and only if $\sum_{n=1}^{\infty} \frac{1}{n\psi(n)}$ converges.

Display (3) implies that $S_N(\alpha) = \mathcal{O}(\log N)$ if $D_N^*(\{n\frac{\alpha}{2}\}) = \mathcal{O}(\frac{\log N}{N})$, but not necessarily vice versa. Since $\sqrt{2}$ and $\sqrt{5} + 1$ are quadratic irrationals, Behnke's result implies that $S_N(\sqrt{2})$ and $S_N(\sqrt{5} + 1)$ have logarithmic bounds, but not that $S_N(\frac{2}{e-1})$ does. Schoißengeier's work [19] can be used to find

$$\limsup_{N \geq 1} \frac{D_N^*(\{n\frac{\alpha}{2}\})}{\log N}$$

for any specific quadratic irrational, but this is typically strictly larger than

$$\limsup_{N \geq 1} \frac{|S_N(\alpha)|}{\log N}.$$

To be fair, we note that the discrepancy approach says that $S_N(\sqrt{3})$ has a logarithmic bound; this *does not* follow from our work in the next sections.

In light of Schimdt's result, it is impossible to use a discrepancy bound to prove a bound on $S_N(\alpha)$ which is sub-logarithmic. As a result of Corollary 3.3 below, however, for any function $\psi(N)$ which increases to infinity with $\psi(1) \geq 1$, we can find an α with $|S_N(\alpha)| \leq \psi(N)$ for all N . For example,

$$S_N\left(\frac{2}{e-1}\right) = \mathcal{O}\left(\frac{\log N}{\log \log N}\right).$$

Khintchine's result and display (3) imply that for almost all real numbers α

$$S_N(\alpha) = \mathcal{O}(\log N \psi(\log N))$$

if $\sum_{n=1}^{\infty} (n\psi(n))^{-1} < \infty$. In particular, set $\psi(n) = \sqrt{n}$ and observe that if $f(N) = \mathcal{O}((\log N)^{3/2})$ then for all large N we have $|f(N)| \leq (\log N)^2$. This verifies the rough statement of display (1). It is conceivable that $S_N(\alpha) = \mathcal{O}(\log N)$ for almost all α , but we find this unlikely.

2 Ideas

The sequence $S_N(\alpha)$ has several near-symmetries, which we formally state and prove as "ideas." In this section, we explicitly state and prove two of them, and show how they can be combined to prove Theorem 2.1. Line (2) is an immediate consequence of Theorem 2.1.

Theorem 2.1. *If m is a positive integer, then*

$$\left| S_N(\sqrt{m^2 + 1} - m + 1) \right| \leq \frac{\log N}{2 \log(\sqrt{m^2 + 1} + m)} + 1.$$

Our first idea is a combination of Beatty's Theorem and a peculiar renormalization of $S_N(\alpha)$. We first state Beatty's Theorem, and give the proof of Ostrowski & Hyslop [1] first published in this journal in 1927. Then we give the renormalization, and then our two ideas.

Lemma 2.2 (Beatty's Theorem). *If $\alpha > 1$ is irrational and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then the sequences $(\lfloor n\alpha \rfloor)$ and $(\lfloor n\beta \rfloor)$ partition \mathbb{N} .*

Proof. Take $N \in \mathbb{N}$; we will show that there is exactly one term of the two sequences $(n\alpha)$ and $(n\beta)$ between N and $N + 1$. Since $N\alpha$ is not an integer, there are $\lfloor N/\alpha \rfloor$ multiples of α less than N , and likewise there are $\lfloor N/\beta \rfloor$ multiples of β . Therefore, from the two sequences there are exactly $\lfloor N/\alpha \rfloor + \lfloor N/\beta \rfloor$ terms strictly less than N . After writing

$$N - 2 = N \left(\frac{1}{\alpha} + \frac{1}{\beta} \right) - 2 = \left(\frac{N}{\alpha} - 1 \right) + \left(\frac{N}{\beta} - 1 \right) < \left\lfloor \frac{N}{\alpha} \right\rfloor + \left\lfloor \frac{N}{\beta} \right\rfloor < \frac{N}{\alpha} + \frac{N}{\beta} = N,$$

we see that $N - 2 < \lfloor N/\alpha \rfloor + \lfloor N/\beta \rfloor < N$. In other words, there are exactly $\lfloor N/\alpha \rfloor + \lfloor N/\beta \rfloor = N - 1$ terms below N . Apply the same reasoning with $N + 1$ in place of N , and we find that there are exactly N terms below $N + 1$. Thus, as claimed, there is exactly one term between N and $N + 1$. \square

And now for our peculiar renormalization. Let

$$S(\alpha; x) := \frac{1}{4} + \sum_{0 < n\alpha \leq x} (-1)^{\lfloor n\alpha \rfloor} = \frac{1}{4} + \sum_{n=1}^{\lfloor x/\alpha \rfloor} (-1)^{\lfloor n\alpha \rfloor}.$$

In terms of our earlier notation, $S_N(\alpha) = S(\alpha; N\alpha) - \frac{1}{4}$. We have achieved two things with this definition. First, we have introduced “ $1/4$ ” at a strategic moment (with 20–20 hindsight). Second, the parameter x is naturally scaled for multiples of α . This is important, since the sums $S(\alpha; x)$ and $S(\beta; x)$ (with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$) are almost complementary. We invite the dubious reader to rewrite Idea 1 in terms of $S_N(\alpha)$.

Idea 1. *If $\alpha > 1$ is irrational and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then $S(\alpha; x) + S(\beta; x) = \pm \frac{1}{2}$.*

Proof. Since α and β satisfy the hypotheses of Beatty’s Theorem, the set of all integers $\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor$ — where $n\alpha$ and $n\beta$ are in $(0, x]$ — will constitute an unbroken block of integers beginning at 1 and continuing either to $\lfloor x \rfloor$ or $\lfloor x \rfloor - 1$. Thus, we write the deepest equation in this section:

$$S(\alpha; x) + S(\beta; x) = \frac{2}{4} + \sum_{j=1}^r (-1)^j$$

for some r . The right-hand side is $\pm \frac{1}{2}$, depending on the parity of r . \square

Our second idea is really little more than the inauspicious observation that $(-1)^2 = 1$.

Idea 2. *If $m \in \mathbb{N}$ and $\beta > 2m$, then $S(\beta; x) = S(\beta - 2m; \frac{\beta - 2m}{\beta}x)$.*

Proof. By hypothesis $\beta > 2m > 0$, so we can multiply inequalities by $\frac{\beta - 2m}{\beta}$:

$$\llbracket 0 < n\beta \leq x \rrbracket = \llbracket 0 < n(\beta - 2m) \leq \frac{\beta - 2m}{\beta}x \rrbracket.$$

Since $\lfloor n(\beta - 2m) \rfloor = \lfloor n\beta \rfloor - 2nm \equiv \lfloor n\beta \rfloor \pmod{2}$, we know that

$$\begin{aligned} S(\beta; x) &= \frac{1}{4} + \sum_{0 < n\beta \leq x} (-1)^{\lfloor n\beta \rfloor} \\ &= \frac{1}{4} + \sum_{0 < n(\beta - 2m) \leq \frac{\beta - 2m}{\beta}x} (-1)^{\lfloor n(\beta - 2m) \rfloor} \\ &= S(\beta - 2m; \frac{\beta - 2m}{\beta}x). \end{aligned}$$

\square

Proof of Theorem 2.1. For the sake of being specific we work with $m = 1$, setting $\alpha := \sqrt{2}$. Afterwards, we indicate the minor changes needed for general m . Since we wish to apply Idea 1, and $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}+2} = 1$, we set $\beta = \sqrt{2} + 2$. In this notation, Idea 1 becomes

$$|S(\alpha; x)| \leq |S(\beta; x)| + \frac{1}{2},$$

and Idea 2 (with $m = 1$) becomes

$$S(\beta; x) = S(\alpha; (\sqrt{2} - 1)x).$$

Set $\lambda = \sqrt{2} - 1$. Combining these two ideas, and iterating, we find

$$\begin{aligned} |S(\alpha; x)| &\leq |S(\beta; x)| + \frac{1}{2} = |S(\alpha; \lambda x)| + \frac{1}{2} \\ &\leq |S(\alpha; \lambda^2 x)| + \frac{1}{2} + \frac{1}{2} \\ &\vdots \\ &\leq |S(\alpha; \lambda^k x)| + \frac{k}{2}. \end{aligned}$$

If $k > \log_\lambda(\frac{\alpha}{x})$, then $\lambda^k x < \alpha$ and so $S(\alpha; \lambda^k x) = \frac{1}{4}$. Setting $k := 1 + \lceil \log_\lambda(\frac{\alpha}{x}) \rceil$ leads us to the inequality

$$|S(\alpha; x)| \leq \frac{1}{4} + \frac{k}{2} = \frac{1}{4} + \frac{1}{2} \left(1 + \left\lfloor \frac{\log(\alpha/x)}{\log \lambda} \right\rfloor \right) \leq \frac{\log(x/\alpha)}{2 \log(\lambda^{-1})} + \frac{3}{4}.$$

Since $|S_N(\alpha)| = |S(\alpha; N\alpha) - \frac{1}{4}| \leq |S(\alpha; N\alpha)| + \frac{1}{4}$,

$$|S_N(\alpha)| \leq \frac{\log N}{2 \log(\lambda^{-1})} + 1$$

as claimed. For general m , set $\alpha_m := \sqrt{m^2 + 1} + m - 1$, $\beta_m := \sqrt{m^2 + 1} + m + 1$ and $\lambda_m := \sqrt{m^2 + 1} - m$. The proof goes through verbatim. \square

3 rd Idea

Our third idea starts from the observation that if α is rational, say $\alpha = p/q$, then $(-1)^{\lfloor n\alpha \rfloor}$ is periodic. In particular, if $N > q$ then

$$\begin{aligned} S_N\left(\frac{p}{q}\right) &= \sum_{n=1}^q (-1)^{\lfloor n \frac{p}{q} \rfloor} + \sum_{n=1}^{N-q} (-1)^{\lfloor (q+n) \frac{p}{q} \rfloor} \\ &= \sum_{n=1}^q (-1)^{\lfloor n \frac{p}{q} \rfloor} + \sum_{n=1}^{N-q} (-1)^{p + \lfloor n \frac{p}{q} \rfloor} = S_q\left(\frac{p}{q}\right) + (-1)^p S_{N-q}\left(\frac{p}{q}\right). \end{aligned}$$

This allows us to replace S_N with the shorter sums S_q and S_{N-q} , and if $N - q > q$, we can replace S_{N-q} with an even shorter sum.

If α is irrational, then $(-1)^{\lfloor n\alpha \rfloor}$ will not be periodic, but if α is “close” to p/q , then $(-1)^{\lfloor n\alpha \rfloor}$ should be “close” to periodic. This is our third idea. Before we state it quantitatively, though, we need to introduce simple continued fractions and the continued fraction expansion. Most books on elementary number theory have a chapter on continued fractions, and the base- α continued fraction expansion of an integer (which we define below) is dealt with in [7, 19], among other places.

3.1 Continued Fractions

Throughout this article, a_i is always an integer, and if $i > 0$ then a_i is positive. We define the function $[\cdot]$ by $[a_0] := a_0$ and for $r \geq 1$

$$[a_0; a_1, a_2, \dots, a_r] := a_0 + \frac{1}{[a_1; a_2, \dots, a_r]} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_r}}}$$

For every rational p/q , there is a unique r and a unique sequence a_0, a_1, \dots, a_r with $a_r \geq 2$ and $\frac{p}{q} = [a_0; a_1, \dots, a_r]$. The limit

$$[a_0; a_1, a_2, \dots] := \lim_{r \rightarrow \infty} [a_0; a_1, a_2, \dots, a_r]$$

always exists and is always irrational. Moreover, for each irrational α , there is a unique sequence a_0, a_1, \dots with $\alpha = [a_0; a_1, \dots]$; we call $[a_0; a_1, \dots]$ the continued fraction of α , and the a_t ($t \geq 0$) are the partial quotients of α .

Given α , one may compute the a_i by noting that $a_0 = [\alpha]$, and inductively using the equation $\alpha = a_0 + [a_1; a_2, \dots]^{-1}$. For example, $[a_1; a_2, \dots] = (\alpha - a_0)^{-1}$, so $a_1 = [(\alpha - a_0)^{-1}]$, and now $[a_2; a_3, \dots] = ((\alpha - a_0)^{-1} - a_1)^{-1}$. The easiest concrete example is $\alpha = \sqrt{2} + 1$. We have $a_0 = [\sqrt{2} + 1] = 2$, and $\sqrt{2} + 1 = 2 + [a_1; a_2, \dots]^{-1}$, whence $[a_1; a_2, \dots] = (\sqrt{2} - 1)^{-1} = \sqrt{2} + 1$ and $a_1 = [\sqrt{2} + 1] = 2$. Iterating, we find $\sqrt{2} + 1 = [2; 2, 2, 2, \dots]$; we note the minor variation

$$\sqrt{2} = [1; 2, 2, 2, \dots]$$

and the generalization

$$\alpha_m := \sqrt{m^2 + 1} + m - 1 = [1; 2m, 2m, 2m, \dots],$$

which we leave as an exercise for the reader. A more difficult example is

$$\frac{2}{e-1} = [1; 6, 10, 14, 18, \dots],$$

which the industrious can derive from the Taylor expansion of e^x at $x = 1$:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2}(1 + \frac{1}{3}(1 + \frac{1}{4}(1 + \dots))).$$

A difficult theorem of Behnke [6, Cor. 1.65] states that $D_N^*(\alpha) = \mathcal{O}(\frac{\log N}{N})$ if and only if the sequence $(\frac{1}{n} \sum_{i=1}^n a_i)$ is bounded, where $\alpha = [a_0; a_1, a_2, \dots]$. This is true of $\alpha_m/2$, and indeed to every quadratic irrational, but is not true of $\frac{1}{e-1}$. In fact, the set of α to which this applies has measure zero, but there are few well-known irrationals for which it is known if the sequence $(\frac{1}{n} \sum_{i=1}^n a_i)$ is bounded. For example, two famous problems are to determine the status of π (see [10]) and $2^{1/3}$ (see [8, Prob. F22]). We showed in Section 1 that $|S_N(\alpha)| \leq 2N D_N^*(\{\frac{\alpha}{2}\})$, so this tells us that $S_N(\alpha) = \mathcal{O}(\log N)$ if the partial quotients of $\frac{\alpha}{2}$ are bounded in average. There are α , however, such that $S_N(\alpha)$ has a logarithmic bound but $2N D_N^*(\{n\frac{\alpha}{2}\})$ does not.

We now inductively define two sequences using the partial quotients of α .

$$\begin{aligned} p_{-2} &= 0, & p_{-1} &= 1, & p_i &= a_i p_{i-1} + p_{i-2}, \\ q_{-2} &= 1, & q_{-1} &= 0, & q_i &= a_i q_{i-1} + q_{i-2}, \end{aligned}$$

The q_i are called the continuants of α . The remarkable (albeit elementary) fact here is that

$$\frac{p_i}{q_i} = [a_0; a_1, a_2, \dots, a_i].$$

The rationals p_i/q_i are called the convergents to α . The general utility of continued fractions lies not in the convergence of the convergents to α , but in the fact that p_i/q_i is the closest rational to α with denominator less than q_{i+1} . This is so important that we state a strong form of this principle explicitly as Lemma 3.1. For a proof, we refer the reader to [9, proof of Thm. 182], [15, Thm. 7.13] or [17, Thm. 10.15].

Lemma 3.1. *If $0 < n < q_{i+1}$, then $|q_i \alpha - p_i| \leq \{n\alpha\} \leq 1 - |q_i \alpha - p_i|$ with equality only if $n = q_i$.*

There are several equivalent definitions of the base- α continued fraction expansion (CFE) of a nonnegative integer N . Fix α , and let q_i denote the continuants of α . We define the CFE of N to be the lexicographically first sequence (Z_i) of nonnegative integers satisfying

$$N = \sum_{i=0}^{\infty} Z_i q_i. \tag{4}$$

In other words, write N as a sum of continuants greedily (i.e., always using the largest possible), and set Z_i to be the number of times you used q_i . The following definitions are equivalent and useful.

- The CFE of 0 is $0, 0, 0, \dots$, and the CFE of $N - q_I$ is

$$Z_0, \dots, Z_{I-1}, Z_I - 1, Z_{I+1}, \dots,$$

where q_I is the largest continuant less than or equal to N .

- The Z_i are nonnegative integers satisfying (4) and $\sum_{i=0}^j Z_i q_i < q_{j+1}$ for every $j \geq 0$.
- The Z_i are integers satisfying (4) and $0 \leq Z_i \leq a_{i+1} - \llbracket Z_{i-1} > 0 \rrbracket$ for every $i \geq 0$.

The continuants q_0, q_1, q_2, \dots are positive and increasing. Therefore, if I is such that $q_I > N$, then $Z_i = 0$ for $i \geq I$. We denote the CFE of N by

$$N = (Z_I, Z_{I-1}, \dots, Z_1, Z_0).$$

Note that we have reversed the order of the Z_i ; this is analogous to the custom of writing $4 + 3 \cdot 10 + 2 \cdot 10^2$ as 234.

We now compute the CFEs of 100 and of 10^{11} with $\alpha = 2/(e-1) = [1; 6, 10, 14, \dots]$. Minding our p 's and q 's, we make the following table.

i	0	1	2	3	4	5	6	7	8
p_i	1	7	71	1001	18089	398959	10391023	312129649	10622799089
q_i	1	6	61	860	15541	342762	8927353	268163352	9126481321

The largest continuant not larger than 100 is 61; continuing greedily we find that $100 = \mathbf{1} \cdot 61 + \mathbf{6} \cdot 6 + \mathbf{3} \cdot 1$. Thus, $100 = (1, 6, 3)$. We also have $100 = \mathbf{16} \cdot 6 + \mathbf{4} \cdot 1$, but $100 \neq (16, 4)$ since this is not a valid CFE (the sequence $16, 4, 0, 0, \dots$ is lexicographically after $1, 6, 3, 0, \dots$). A bit more arithmetic reveals that

$$10^{11} = (10, 32, 17, 6, 8, 15, 11, 9, 0),$$

a fact which we make use of in another example below.

3.2 The Idea

Idea 3. If p/q is the convergent to α with $q < N$ maximal, then

$$S_N(\alpha) = S_q(\alpha) + (-1)^p S_{N-q}(\alpha).$$

Proof. Define I by $q = q_I$. We show that $\lfloor \{n\alpha\} + q\alpha - p \rfloor = 0$ for $0 < n < q_{I+1}$ by showing that

$$0 < \{n\alpha\} + q\alpha - p < 1 \quad \text{for} \quad 0 < n < q_{I+1}. \quad (5)$$

If $n \neq q$, then from Lemma 3.1 we know that

$$-(q\alpha - p) \leq |q\alpha - p| < \{n\alpha\} < 1 - |q\alpha - p| \leq 1 - (q\alpha - p),$$

which verifies the inequalities in Line (5). If $n = q$, then either $q\alpha - p = \{n\alpha\}$ or $q\alpha - p = \{n\alpha\} - 1$. In the first case, $0 < q\alpha - p < \frac{1}{2}$ and $\{n\alpha\} + q\alpha - p = 2(q\alpha - p) \in (0, 1)$; in the second case, $-\frac{1}{2} < q\alpha - p < 0$ and $\{n\alpha\} + q\alpha - p = 2(q\alpha - p) + 1 \in (0, 1)$. Either way, we have verified the inequalities in Line (5).

Since $N - q < N \leq q_{I+1}$, we have

$$\begin{aligned} S_N(\alpha) - S_q(\alpha) &= \sum_{n=1}^{N-q} (-1)^{\lfloor (n+q)\alpha \rfloor} \\ &= \sum_{n=1}^{N-q} (-1)^{\lfloor \{n\alpha\} + \{n\alpha\} + q\alpha - p + p \rfloor} \\ &= \sum_{n=1}^{N-q} (-1)^{\lfloor n\alpha \rfloor} (-1)^{\lfloor \{n\alpha\} + q\alpha - p \rfloor} (-1)^p \\ &= (-1)^p \sum_{k=1}^{N-q} (-1)^{\lfloor n\alpha \rfloor} \\ &= (-1)^p S_{N-q}(\alpha). \end{aligned}$$

□

3.3 A Formula for $S_N(\alpha)$

We are in position to prove the following theorem. First we state it, then we give an example of it, and then we prove it. In the next subsection, we use Theorem 3.2 to describe the record-holders $R_k(\alpha)$ for those α satisfying the hypothesis of Theorem 3.2.

Theorem 3.2. Let $\alpha = [a_0; a_1, a_2, \dots]$, with a_0 odd and all other a_i even. If $N = (Z_I, Z_{I-1}, \dots, Z_1, Z_0)$, then

$$S_N(\alpha) = - \sum_{i=0}^I \left(\prod_{t=i+1}^I (-1)^{Z_t} \right) \llbracket i \text{ even} \rrbracket \llbracket Z_i \text{ odd} \rrbracket.$$

We note that this formula can be quickly pulled out of Brown's decomposition (which we won't state) of Sturmian words (which we won't define); indeed, this is how the authors first found Theorem 3.2. A similar result, but applying only to $\sqrt{2}$, appears in [5, Section 5]; their proof uses ideas similar to those used by Brown [4]. Our proof of Theorem 3.2 borrows from the proof of Brown's decomposition given by the first author in [16].

An example will illustrate the power this result harnesses. We will compute $S_N(\alpha)$ with $N = 10^{11}$ and $\alpha = 2/(e-1)$. We noted above that

$$10^{11} = (10, 32, 17, 6, 8, 15, 11, 9, 0),$$

and now note that Z_i is odd only for $i \in \{1, 2, 3, 6\}$. Thus

$$\llbracket i \text{ even} \rrbracket \llbracket Z_i \text{ odd} \rrbracket = \llbracket i \in \{2, 6\} \rrbracket.$$

We apply Theorem 3.2 to get

$$\begin{aligned} S_{100000000000} \left(\frac{2}{e-1} \right) &= - \sum_{i=0}^8 \left(\prod_{t=i+1}^8 (-1)^{Z_t} \right) \llbracket i \text{ even} \rrbracket \llbracket Z_i \text{ odd} \rrbracket \\ &= - \left(\prod_{t=2+1}^8 (-1)^{Z_t} + \prod_{t=6+1}^8 (-1)^{Z_t} \right) \\ &= - \left((-1)^{15+8+6+17+32+10} + (-1)^{32+10} \right) \\ &= -2. \end{aligned}$$

A 1 Ghz PC running Mathematica would take about 20 days to compute this naively.

Proof of Theorem 3.2. Since α is fixed throughout this proof, and to avoid having subscripts with subscripts, we set $S(N) := S_N(\alpha)$. By definition $p_0 = a_0$, $p_1 = a_0 a_1 + 1$, and $p_i = a_i p_{i-1} + p_{i-2}$; we find that all p_i are odd. Idea 3 tells us that

$$S(q_i) = S(q_{i-1}) - S(q_i - q_{i-1}) = S(q_{i-1}) - (S(q_{i-1}) - S(q_i - 2q_{i-1})) = S(q_i - 2q_{i-1})$$

whence

$$S(q_i) = S(q_i - 2q_{i-1}) = S(q_i - 4q_{i-1}) = \dots = S(q_i - a_i q_{i-1}) = S(q_{i-2})$$

and so

$$S(q_i) = S(q_{i-2}) = S(q_{i-4}) = \dots = \begin{cases} S(q_{-1}), & i \text{ odd;} \\ S(q_0), & i \text{ even.} \end{cases}$$

Now

$$S(q_{-1}) = S(0) = 0$$

and

$$S(q_0) = S(1) = (-1)^{\lfloor \alpha \rfloor} = (-1)^{a_0} = -1,$$

which completely solves the problem when N is a denominator of a convergent to α . Note that this gives

$$S(q_i) = -\llbracket i \text{ even} \rrbracket$$

which matches with the formula stated in the theorem. This will serve both as a basis for induction, and as a key step in the induction itself.

Suppose that the formula has been proved for all arguments less than $N = (Z_I, \dots, Z_0)$, with I chosen so that $Z_I \neq 0$. We have two cases; either $Z_I > 1$ or $Z_I = 1$.

If $Z_I > 1$ (the easy case, so we do it first), then applying Idea 3 twice yields (writing q in place of q_I)

$$S(N) = S(q) - S(N - q) = S(q) - (S(q) - S(N - 2q)) = S(N - 2q).$$

Since the formula we are proving cares only about the parity of Z_i , we need only to apply the induction hypothesis to $N - 2q_I = (Z_I - 2, Z_{I-1}, \dots, Z_0)$ to complete the proof.

If $Z_I = 1$, then we have (with $q = q_I$)

$$\begin{aligned} S(N) &= S(q) - S((Z_{I-1}, \dots, Z_0)) \\ &= S(q) + \sum_{i=0}^{I-1} \left(\prod_{t=i+1}^{I-1} (-1)^{Z_t} \right) \llbracket i \text{ even} \rrbracket \llbracket Z_i \text{ odd} \rrbracket. \end{aligned} \quad (6)$$

Note that we have used the induction hypothesis applied to (Z_{I-1}, \dots, Z_0) , and the fact that the formula is not affected if we pad the left of the CFE with zeros (since it may happen that $Z_{I-1} = 0$, for example). We can expand the $S(q_I)$ term by observing that $S(q) = -\llbracket I \text{ even} \rrbracket$ by our base case, $\llbracket Z_I \text{ odd} \rrbracket = 1$ since $Z_I = 1$, and $\prod_{t=I+1}^I (-1)^{Z_t} = 1$ since it is an empty product. Thus,

$$S(q) = - \left(\prod_{t=I+1}^I (-1)^{Z_t} \right) \llbracket I \text{ even} \rrbracket \llbracket Z_I \text{ odd} \rrbracket.$$

Further, $\prod_{t=i+1}^{I-1} (-1)^{Z_t} = - \prod_{t=i+1}^I (-1)^{Z_t}$ for $0 \leq i < I$ since $Z_I = 1$, so that

$$\sum_{i=0}^{I-1} \left(\prod_{t=i+1}^{I-1} (-1)^{Z_t} \right) \llbracket i \text{ even} \rrbracket \llbracket Z_i \text{ odd} \rrbracket = - \sum_{i=0}^{I-1} \left(\prod_{t=i+1}^I (-1)^{Z_t} \right) \llbracket i \text{ even} \rrbracket \llbracket Z_i \text{ odd} \rrbracket.$$

Line (6) now reduces to the formula as stated in the theorem. \square

3.4 The Record Holders

Recall that the record-holder at k is defined by

$$R_k(\alpha) := \min\{N \geq 0: S_N(\alpha) = k\}.$$

Obviously $R_0(\alpha) = 0$; the following corollary considers $k \neq 0$ and α satisfying the hypotheses of Theorem 3.2.

Corollary 3.3. *Let $\alpha = [a_0; a_1, a_2, \dots]$, with a_0 odd and all other a_i even, and let q_i be the continuants. Then*

$$R_k(\alpha) = q_{2k-1} \llbracket k > 0 \rrbracket + \sum_{i=0}^{2|k|-2} q_i.$$

In particular,

$$R_k(\sqrt{2}) = \begin{cases} \lfloor \frac{1}{4}(\sqrt{2} + 1)^{2k+1} \rfloor, & k > 0; \\ \lfloor \frac{1}{4}(\sqrt{2} + 1)^{-2k} \rfloor, & k < 0. \end{cases}$$

Proof. We briefly indicate the $k < 0$ case; the “positive k ” case is similar and left to our most diligent readers. Set

$$N_k := \sum_{i=0}^{2|k|-2} q_i$$

It is not obvious that the CFE of N_k is $(1, 1, \dots, 1)$; to show this one must show that $\sum_{i=0}^j q_i < q_{j+1}$ for all $j \geq 0$, which requires use of the hypothesis that $a_i \geq 2$ (for $i > 0$).

Assuming this, we have $Z_i = 1$ for $0 \leq i \leq 2|k| - 2$, and

$$\prod_{t=i+1}^{2|k|-2} (-1)^{Z_t} \llbracket i \text{ even} \rrbracket = (-1)^{2|k|-2-i} \llbracket i \text{ even} \rrbracket = \llbracket i \text{ even} \rrbracket.$$

In light of this, Theorem 3.2 reduces to

$$S_N(\alpha) = - \sum_{i=0}^{2|k|-2} \llbracket i \text{ even} \rrbracket = k,$$

so that $R_k(\alpha) \leq N_k$.

Now suppose that $R_k(\alpha) = M = (Z_{I'}, Z_{I'-1}, \dots, Z_0) \leq N_k$, where $I' \leq 2|k| - 2$ since $M \leq N_k$. By taking the absolute value of both sides of the formula given in Theorem 3.2 and using $\llbracket Z_i \text{ odd} \rrbracket \leq 1$, we find that

$$|k| = |S_M(\alpha)| \leq \sum_{i=0}^{I'} \llbracket i \text{ even} \rrbracket = \left\lfloor \frac{I'}{2} \right\rfloor + 1 \leq \frac{I'}{2} + 1,$$

and consequently that $I' \geq 2|k| - 2$. Thus, $I' = 2|k| - 2$, and Theorem 3.2 now gives

$$\begin{aligned} k &= - \sum_{i=0}^{2|k|-2} \left(\prod_{t=i+1}^{2|k|-2} (-1)^{Z_t} \right) \llbracket i \text{ even} \rrbracket \llbracket Z_i \text{ odd} \rrbracket \\ &= - \sum_{s=0}^{|k|-1} \left(\prod_{t=2s+1}^{2|k|-2} (-1)^{Z_t} \right) \llbracket Z_{2s} \text{ odd} \rrbracket. \end{aligned}$$

The sum over s has $|k|$ terms, each with absolute value 0 or 1, and the sum is k . Therefore,

$$1 = \llbracket Z_{2s} \text{ odd} \rrbracket \prod_{i=2s+1}^{2|k|-2} (-1)^{Z_i}$$

for $0 \leq s \leq |k| - 1$. For $s = |k| - 1$, this implies that $Z_{2|k|-2}$ and $Z_{2|k|-1}$ are both odd, and taking successively smaller values for s informs us that all Z_i are odd ($0 \leq i \leq 2|k| - 2$). Thus $M \geq \sum_{i=0}^{2|k|-2} q_i = N_k$.

For $\alpha = \sqrt{2}$, the continuants are defined by $q_0 = 1$, $q_1 = 2$, and $q_i = 2q_{i-1} + q_{i-2}$. Solving this recurrence gives

$$q_i = \frac{(1 + \sqrt{2})^{i+1} - (1 - \sqrt{2})^{i+1}}{2\sqrt{2}}$$

and

$$N_k := \sum_{i=0}^{2|k|-2} q_i = -\frac{1}{2} + \frac{(1 + \sqrt{2})^{-2k}}{4} + \frac{(1 - \sqrt{2})^{-2k}}{4} = \left\lfloor \frac{(1 + \sqrt{2})^{-2k}}{4} \right\rfloor,$$

establishing the $k < 0$ part of the ‘in particular’ part of the corollary. \square

The next corollary, our finale, is in pleasant contrast to Theorem 2.1.

Corollary 3.4. *There are infinitely many N with*

$$|S_N(\sqrt{2})| > \frac{\log N}{2\log(1 + \sqrt{2})} + \frac{\log 2}{\log(1 + \sqrt{2})} > \frac{\log N}{2\log(1 + \sqrt{2})} + 0.78.$$

Proof. Take $k < 0$, and set $N := R_k(\sqrt{2}) < \frac{1}{4}(1 + \sqrt{2})^{2|k|}$. Solving for $|k|$, we find

$$|S_N(\alpha)| = |k| > \frac{\log(4N)}{2\log(\sqrt{2} + 1)}.$$

\square

4 Harder Questions

We now state four problems that the authors have been unable to resolve.

- What happens with other α ? For example, we know from Section 1 that if α is any quadratic irrational, then $S_N(\alpha) = \mathcal{O}(\log N)$. Is this the correct type of growth for all quadratic irrationals? What are the necessary and sufficient conditions on α for $S_N(\alpha) = \mathcal{O}(\log N)$, in terms of the continued fraction expansion of α ? It seems unlikely that $S_N(\alpha)$ is $\mathcal{O}(\log N)$ for almost all α , but a proof of this is elusive.
- Given a base m , what can be said about the number of $[n\alpha]$ ($1 \leq n \leq N$) which belong to the various congruence classes modulo m ? There are $m - 1$ free parameters here, because the sum will always be N . For $m = 2$, this is an equivalent problem to the one we have worked on in this article; for $m \geq 3$ it is different. As a starting point, one may wish to consider the sequence

$$\sum_{n=1}^N (e^{2\pi i/m})^{[n\alpha]}.$$

- Our ideas do not generalize in any really straightforward manner to handle nonhomogeneous sequences. For example, is

$$\left| \sum_{n=1}^N (-1)^{\lfloor n\sqrt{2} + \frac{1}{2} \rfloor} \right| \leq \frac{\log N}{2\log(1 + \sqrt{2})} + \frac{3}{2}$$

for $N > 2$?

- The problem of bounding $S_N(\sqrt{2})$ originally arose in studying the convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^{\lfloor n\sqrt{2} \rfloor}}{n}.$$

Now that we know sharp bounds for $|S_N(\sqrt{2})|$, is it possible to evaluate this infinite sum? Is it rational?

Acknowledgement

The first author thanks the National Science Foundation for their support. All the authors thank Dennis Eichhorn for giving the manuscript a careful proofreading.

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