

A quantitative Pólya's Theorem with corner zeros

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Abstract

Pólya's Theorem says that if p is a homogeneous polynomial in n variables which is positive on the standard n -simplex, and F is the sum of the variables, then for a sufficiently large exponent N , $F^N * p$ has positive coefficients. Pólya's Theorem has had many applications in both pure and applied mathematics, for example it provides a certificate for the positivity of p on the simplex. The authors have previously given an explicit bound on N , determined by the data of p ; namely, the degree, the size of the coefficients and the minimum value of p on the simplex. In this paper, we extend this quantitative Pólya's Theorem to non-negative polynomials which are allowed to have simple zeros at the corners of the simplex.

1 Introduction

Fix a positive integer n and let $\mathbb{R}[X] := \mathbb{R}[x_1, \dots, x_n]$. We use the following polynomial notation: Given $\alpha \in \mathbb{N}^n$, say $\alpha = (\alpha_1, \dots, \alpha_n)$, we denote by X^α the monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and write $|\alpha|$ for $\alpha_1 + \dots + \alpha_n$. For $u = (u_1, \dots, u_n) \in \mathbb{R}^n$, we similarly write u^α for $u_1^{\alpha_1} \dots u_n^{\alpha_n} \in \mathbb{R}$. We denote the standard n -simplex $\{(x_1, \dots, x_n) \mid x_i \geq 0, \sum_i x_i = 1\}$ by Δ_n .

Pólya's Theorem says that if a form (homogeneous polynomial) $p \in \mathbb{R}[X]$ is positive on Δ_n , then for sufficiently large N all the coefficients of

$$(x_1 + \dots + x_n)^N * p(x_1, \dots, x_n)$$

are positive. This elegant and beautiful result has many applications, both in pure and applied mathematics.

Pólya's theorem appeared in 1928 [10] (in German) and is also in *Inequalities* by Hardy, Littlewood, and Pólya [8] (in English). It is interesting to note that they realized the algorithmic nature of the result; in the book they write

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The theorem gives a systematic process for deciding whether a given form F is strictly positive for positive x . We multiply repeatedly by $\sum x_i$ and, if the form is positive, we shall sooner or later obtain a form with positive coefficients.

Pólya's proof is elementary: for p positive on Δ_n of degree d , he constructs a sequence of real polynomials p_ϵ in n variables such that p_ϵ converges uniformly to p on Δ_n as $\epsilon \rightarrow 0$ and so that for $|\beta| = N + d$, the coefficient of X^β in $(\sum x_i)^N * p$ is a positive multiple of $p_\epsilon(\frac{\beta}{N+d})$ for $\epsilon = \frac{1}{N+d}$. Then, since $p_\epsilon > 0$ for sufficiently small ϵ , it follows that for sufficiently large N , each coefficient of X^β is positive.

By analyzing Pólya's proof, the authors in [11] were able to give an explicit bound on the N needed in the theorem. If $|\alpha| = d$, define $c(\alpha) := \frac{d!}{\alpha_1! \dots \alpha_n!}$. Suppose $f \in \mathbb{R}[X]$ is homogeneous of degree d , then write

$$f(X) = \sum_{|\alpha|=d} a_\alpha X^\alpha = \sum_{|\alpha|=d} c(\alpha) b_\alpha X^\alpha,$$

and let $L = L(f) := \max_{|\alpha|=d} |b_\alpha|$ and $\lambda = \lambda(f) := \min_{X \in \Delta_n} f(X)$.

Theorem 1. *Suppose that $f \in \mathbb{R}[X]$ is as given as above. If*

$$N > \frac{d(d-1)L}{2\lambda} - d,$$

then $(x_1 + \dots + x_n)^N f(x_1, \dots, x_n)$ has positive coefficients.

We describe a few applications of Pólya's Theorem and this bound. In 1940, Habicht [5] used Pólya's Theorem to give a direct proof of a special case of Hilbert's 17th Problem, namely, he used it to prove that a positive definite form is a sum of squares of rational functions. More recently, M. Schweighofer [12] used Pólya's Theorem to give an algorithmic proof of Schmüdgen's Positivstellensatz, which says that if the basic closed semialgebraic set $K = \{g_1 \geq 0, \dots, g_k \geq 0\}$ is compact and $f > 0$ on K , then f is in the preorder generated by the g_i 's. This can be used to give an algorithm for optimization of polynomials on compact semialgebraic sets, see [13] for details. Using the bound for Pólya's Theorem, Schweighofer obtained complexity bounds for Schmüdgen's Positivstellensatz [14].

Pólya's Theorem has been used in the study of copositive programming. Let \mathbb{S}^n denote the $n \times n$ symmetric matrices over \mathbb{R} and define the copositive cone

$$C_n = \{M \in \mathbb{S}^n \mid Y^T M Y \geq 0 \text{ for all } Y \in \mathbb{R}_+^n\}.$$

Copositive programming is optimization over C_n . By Pólya's Theorem, the truncated cones

$$C_n^r := \{M \in \mathbb{S}^n \mid (\sum_i x_i)^r * X^T M X \text{ has non-negative coefficients} \}$$

will converge to C_n and using linear programming, membership in C_n^r can be determined numerically. De Klerk and Pasechnik [4] use this fact, along with the bound for Pólya's Theorem, to give results on approximating the stability number of a graph.

Motzkin and Strauss [9] partially generalized the theorem to power sequences in several variables and Catlin and D'Angelo [1, 2] generalized the theorem to polynomials in several complex variables. Handelman [6, 7] has studied a related question, namely, for which pairs (q, f) of polynomials does there exist $N \in \mathbb{N}$ so that $q^N * f$ has nonnegative coefficients? (See also de Angelis and Tuncel [3].) Pólya's Theorem and the generalization described in this paper (without the bound) can be deduced from Handelman's work. It also follows (again, without the bound) from recent work of Schweighofer [15].

In this paper we discuss an extension of Pólya's Theorem to the case where the form p has zeros on Δ_n . By methods similar to those used in [11] to prove Theorem 1, we give a bound on the N needed in the case where p is positive on Δ_n except for possible zeros at the "corners" of the simplex; the bound is in terms of information about the coefficients of p and the minimum of p on Δ_n away from the zeros.

Remark 1. By "positive coefficients" we mean that every coefficient that is non-zero is positive. If a form p is strictly positive on Δ_n , then Pólya's Theorem shows that there is N so that every monomial in $(x_1 + \dots + x_n)^N p$ has a positive coefficient. However this will not be possible if p has zeros on Δ_n , e.g., if $p(1, 0, \dots, 0) = 0$ then the coefficient of x_1^{N+d} in $(x_1 + \dots + x_n)^N p$ will always be zero.

2 Pólya's Theorem for forms non-negative on the simplex

Let $P_{n,d}(\Delta_n)$ denote the closed cone of degree d forms in n variables which are non-negative on Δ_n and let $Po(n, d)$ be the degree d forms in n variables which satisfy Pólya's Theorem, i.e., $p \in Po(n, d)$ if there is some N such that every monomial in $(x_1 + \dots + x_n)^N p$ has a positive coefficient. For ease of exposition, denote the form $x_1 + \dots + x_n$ by F .

Given $p \in \mathbb{R}[X]$ of degree d , write

$$p = \sum_{|\alpha| \leq d} a_\alpha X^\alpha;$$

we let $\text{supp}(p)$ denote $\{\alpha \mid a_\alpha \neq 0\}$. We write e_1, \dots, e_n for the vertices of Δ_n , i.e., $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$.

Suppose $p \in P_{n,d}(\Delta_n)$ and p has a zero on the interior of Δ_n , say $p(u) = 0$ for $u = (u_1, \dots, u_n) \in \Delta_n$ with $u_i > 0$ for all i . Then it is not too hard to see that $p \notin Po(n, d)$. For every $N \in \mathbb{N}$, if $F^N p = \sum c_\alpha X^\alpha$, then $0 = \sum c_\alpha u^\alpha$ and $u^\alpha > 0$ for all α , hence at least one c_α must be negative. On the other hand, $p = x_1 \dots x_n$

is trivially in $Po(n, d)$ and has $p(u) = 0$ for every u on the boundary of Δ_n . Also, if p vanishes at an interior point of a face of Δ_n , then p vanishes everywhere on the face and has a monomial factor, hence it makes sense to restrict our attention to zeros on faces of co-dimension at least 2.

We note also that it is possible to have $p \in Po(n, d)$, but not in $Po(n + 1, d)$ when p is considered as a form in $n + 1$ variables. Let $p = x^2 - xy + y^2$, then $(x + y)p = x^3 + y^3$ and hence $p \in Po(2, 2)$. However, for every N , the coefficient of $x^1 y^1 z^N$ in $(x + y + z)^N (x^2 - xy + y^2)$ is -1 , thus $p \notin Po(3, 2)$.

Example 1. The following forms are non-negative on Δ_3 with zeros only at the unit vectors:

$$\begin{aligned} f &= xz^3 + yz^3 + x^2y^2 - xyz^2, \\ g &= x^2y + xy^2 - 3xyz. \end{aligned}$$

We claim $f \notin Po(3, 3)$, but $g \in Po(3, 3)$. Consider the coefficient of $x^{N+1}yz^2$ in $F^N * f$. There is no contribution from $F^N xz^3$ or $F^N yz^3$ because the power of z is too large and there is no contribution from $F^N x^2y^2$ because the power of y is too large. Hence the only contribution comes from $F^N (-xyz^2)$ and thus the coefficient will always be -1 . On the other hand, it is easy to compute that $F^3 g$ has only positive coefficients. This shows that the location of the zeros of $p \in P_{n,d}(\Delta_n)$ is not enough to determine if p is in $Po(n, d)$ or not.

Definition 1. The form $p \in P_{n,d}(\Delta_n)$ has a *simple zero* at the unit vector e_j if the coefficient of x_j^d in p is zero, but the coefficient of $x_j^{d-1}x_i$ is non-zero (and necessarily positive) for each $i \neq j$. In other words, $\text{supp}(p)$ contains $(d-1) \cdot e_j + e_i$ for $i \neq j$, but not $d \cdot e_j$. Geometrically, this means that when p is restricted to lines through e_j and another point in Δ_n , it has only a simple zero at e_j .

For $r \in \mathbb{R}$, $0 < r < 1$ and $j = 1, \dots, n$, let $\Delta_n(j, r)$ denote the simplex with vertices $\{e_j\} \cup \{e_j + r(e_i - e_j) \mid i \neq j\}$. For example, $D_3(2, r)$ is the triangle with vertices $\{(0, 1, 0), (r, 1 - r, 0), (0, 1 - r, r)\}$. Hence $\Delta_n(j, r)$ is the scaled simplex $r \cdot \Delta_n$ translated by $(1 - r)e_j$.

Lemma 1. *If $p \in P_{n,d}(\Delta_n)$ has a simple zero at e_j , then there exist $s, r > 0$ such that*

$$p(u_1, \dots, u_n) \geq s(u_1 + \dots + u_{j-1} + u_{j+1} + \dots + u_n) \quad (1)$$

for all $u = (u_1, \dots, u_n) \in \Delta_n(j, r)$.

Proof. For $i \neq j$, let $v_i = \frac{\partial p}{\partial x_i}(e_j)$, which is the coefficient of $x_j^{d-1}x_i$. By assumption, $v_i > 0$. It follows from the Taylor series of p at e_j that for $t = (t_1, \dots, t_n)$,

$$p(t_1, \dots, t_{j-1}, 1, t_{j+1}, \dots, t_n) = \sum_i v_i t_i + o(|t|).$$

Let $v = \frac{1}{2} \min v_j$, then for sufficiently small t we have

$$p(t_1, \dots, t_{j-1}, 1, t_{j+1}, \dots, t_n) \geq v \cdot \sum_{i \neq j} t_i.$$

If $u \in \Delta_n(j, r)$, then by homogeneity $p(u) = x_j^d p(t_1, \dots, t_{j-1}, 1, t_{j+1}, \dots, t_n)$ where $t_i := x_i/x_j$ for $i \neq j$. Since $x_j^d \geq (1-r)^d$, we can take $s = v(1-r)^d$. \square

Our goal is to find a quantitative version of Pólya's Theorem which applies to polynomials which are positive on Δ_n , except for some simple zeros at the vertices. We begin with the case of a single simple zero, which we take at e_1 , without loss of generality.

We fix some notation. Suppose $p \in P_{n,d}(\Delta_n)$ is positive on Δ_n except for a simple zero at e_1 . Let $s, r > 0$ be as in Lemma 1 and define the following constants associated to p . Let P be the closure of Δ_n with the corner $\Delta_n(1, r)$ removed and let λ be the minimum of p on P . Define M to be the size of the largest coefficient of p , i.e., $M := \max\{|a_\alpha| \mid \alpha \in \text{supp}(p)\}$ and set $L = L(p)$. Finally, for $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ define

$$c(\beta) := \prod_{\beta_i \geq 2} \frac{\beta_i(\beta_i-1)}{2} \text{ and } c := \sum_{\alpha \in \text{supp}(p)} c(\alpha).$$

Proposition 1. *Suppose p is positive on Δ_n except for a simple zero at e_1 . With the notation as above, if*

$$N > \max\left(\frac{d(d-1)L}{2} \frac{1}{\lambda} - d, \frac{cM}{s} - d\right),$$

then $F^N * p$ has positive coefficients, and hence $p \in Po(n, d)$.

Proof. We proceed as in [10] and [11]. For positive $t \in \mathbb{R}$, $m \in \mathbb{N}$ and a single real variable y , define

$$(y)_t^m := y(y-t) \cdots (y-(m-1)t) = \prod_{i=0}^{m-1} (y-it).$$

and

$$p_t(u_1, \dots, u_n) := \sum_{\alpha \in \text{supp}(p)} a_\alpha (x_1)_t^{\alpha_1} \cdots (x_n)_t^{\alpha_n}.$$

For ease of exposition, set $t = \frac{1}{N+d}$. Then we have

$$A_\beta = \frac{N!(N+d)^d}{\beta_1! \cdots \beta_n!} p_t\left(\frac{\beta_1}{N+d}, \dots, \frac{\beta_n}{N+d}\right), \quad (2)$$

Thus to show $A_\beta > 0$, we need

$$p_t\left(\frac{\beta_1}{N+d}, \dots, \frac{\beta_n}{N+d}\right) > 0.$$

Let $z = (\beta_1 t, \dots, \beta_n t) \in \Delta_n$. Assume $z \in P$, then the proof of Theorem 1 in [11] shows that for $N > \frac{d(d-1)L}{2} \frac{1}{\lambda} - d$, $p_t(z) > 0$ and hence $A_\beta > 0$.

Now suppose $z \in \Delta_n(1, r)$ and let $N = \frac{cM}{s} - d$. Our goal is to show that $p_t(z) > 0$. As in [11], we write

$$p_t(z) = p(z) - \sum_{\alpha \in \text{supp}(p)} a_\alpha (z^\alpha - (z_1)_t^{\alpha_1} \dots (z_n)_t^{\alpha_n}) \quad (3)$$

We have the bound

$$p(z) \geq s(z_2 + \dots + z_n), \quad (4)$$

so we consider the summation in (3). As in the proof of Theorem 1 in [11], it is easy to see that for $0 \leq y \leq 1$ and $k \geq 2$,

$$(y)_t^k \geq y^k - \frac{k(k-1)}{2} t y^{k-1}, \quad (5)$$

this also holds for $k = 1$ trivially. Then, using (5), we have

$$\begin{aligned} & \left| \sum_{\alpha \in \text{supp}(p)} a_\alpha \left(z^\alpha - \left(\prod_{i=1}^n (z_i)_t^{\alpha_i} \right) \right) \right| < \\ & M \left| \sum_{\alpha \in \text{supp}(p)} \left(z^\alpha - \prod_{i=1}^n \left(z_i^{\alpha_i} - \frac{\alpha_i(\alpha_i-1)}{2} \cdot t \cdot z_i^{\alpha_i-1} \right) \right) \right| = \\ & M \left| \sum_{\alpha \in \text{supp}(p)} \left[z^\alpha - z^\alpha \prod_{i=1}^n \left(1 - \frac{\alpha_i(\alpha_i-1)}{2} \cdot t \cdot \frac{1}{z_i} \right) \right] \right| = \\ & M \left| \sum_{\alpha \in \text{supp}(p)} \left[z^\alpha \cdot \left(\prod_{\alpha_i \geq 2} \frac{\alpha_i(\alpha_i-1)}{2} \cdot t \cdot \frac{1}{z_i} \right) \right] \right| \end{aligned}$$

Now, because p has a simple zero at e_1 , for every $\alpha \in \text{supp}(p)$ and every i with $\alpha_i \geq 2$, the monomial z^α/z_i contains at least one of $\{z_2, \dots, z_n\}$. It follows that

$$M \left| \sum_{\alpha \in \text{supp}(p)} \left(z^\alpha \cdot \prod_{\alpha_i \geq 2} \frac{\alpha_i(\alpha_i-1)}{2} \cdot t \cdot \frac{1}{z_i} \right) \right| \leq \sum_{\alpha \in \text{supp}(p)} c(\alpha) \cdot t \cdot (z_2 + \dots + z_n),$$

recalling that $c(\alpha) = \prod_{\alpha_i \geq 2} \frac{\alpha_i(\alpha_i-1)}{2}$.

Combining this with (3) and (4), we have

$$p_t(z) > \left(s - \frac{M}{N+d} \sum_{\alpha \in \text{supp}(p)} c(\alpha) \right) (z_2 + \dots + z_n) > \left(s - \frac{Mc}{N+d} \right) (z_2 + \dots + z_n).$$

Since $z_2 + \dots + z_n > 0$ and $s - \frac{Mc}{N+d} > 0$, we conclude that $p_t(z) > 0$. \square

Remark 2. We note that the bound in Proposition 1 does not depend on n , the number of variables. Also, observe that as $r \rightarrow 0$, $\lambda \rightarrow \infty$ because p has a zero at e_1 . On the other hand, s is bounded and thus the choice of r is more important to the bound from the main part of Δ_n than the bound from the corner.

Corollary 1. *Suppose $p \in P_{n,d}(\Delta_n)$ is positive on Δ_n except for simple zeros at unit vectors e_{j_1}, \dots, e_{j_k} . Then $p \in Po(n, d)$ and there is a bound for N so that $F^N p$ has only positive coefficients similar to the bound in Proposition 1.*

Note that in the Corollary, the simplex P is replaced by the unit simplex with k corners snipped off and s will be replaced by the minimum of the s 's obtained by applying Lemma 1 to each simple zero.

Example 2. For $0 < \alpha < 6$, let

$$p_\alpha(x, y, z) := x(y - z)^2 + y(x - z)^2 + z(x - y)^2 + \alpha xyz.$$

Note that the first three terms give a form non-negative on D_3 with zeros at the unit vectors and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, thus p_α is psd on D_3 with zeros at the unit vectors and is symmetric in $\{x, y, z\}$. We will compute the bound from the proposition, directly compute the minimum N so that $F^N * p$ has positive coefficients, and compare the two.

We start by assuming we have r as in Lemma 1 and compute the minimum of p on the simplex P ; recall that P is D_3 minus the three corners $D_3(j, r)$. We are interested in the behavior as $\alpha \rightarrow 0$, so we may as well assume $\alpha < 1$. It is a straightforward calculus exercise to compute the minimum of p on P : the interior extreme values occur at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and all three permutations of $(\frac{1}{9-\alpha}, \frac{1}{9-\alpha}, \frac{7-\alpha}{9-\alpha})$. Assuming $\alpha < 1$, the smallest value of p_α occurs at the centroid and equals $\frac{\alpha}{27}$. Finding the minimum on the boundary of P involves two line segments, one from $(r, 1-r, 0)$ to $(1-r, r, 0)$, and the other from $(1-r, r, 0)$ to $(1-r, 0, r)$. In the first case, $z = 0$ and the minimum is clearly $r(1-r)$. In the second case, an easy calculation shows that the minimum occurs at $(1-r, \frac{r}{2}, \frac{r}{2})$; the exact value is $r - 3r^2 + \frac{r^2}{4}(\alpha(1-r) + 9r) > r - 3r^2$. If we assume r small enough, say $r \leq \frac{\alpha}{18}$, then the minimum value of p on P is $\lambda = \frac{\alpha}{27}$. The remaining constants are $L = \min\{1, \frac{6-\alpha}{6}\} = 1$ and $d = 3$ and so one bound is equal to $\frac{81}{\alpha} - 3$.

We now compute the bound on the corners. We have

$$p(1, y, z) = y + z + y^2 - (6 - \alpha)yz + z^2 + y^2z + yz^2.$$

Observe that $y^2 - (6 - \alpha)yz + z^2 + (y + z)^2 = 2(y - z)^2 + \alpha yz$ is positive when y and z are positive and thus

$$p(1, y, z) \geq y + z - (y + z)^2 = (y + z)(1 - (y + z)).$$

Hence we can take $s = (1 - r)^3 * (1 - r) = (1 - r)^4$. We have $M = 6 - \alpha$ and $v = 6$. The second bound is therefore

$$\frac{6(6 - \alpha)}{(1 - r)^4 - 3} > \frac{36}{(1 - r)^4} - 3.$$

Putting this together, if r is chosen so that $r - 3r^2 > \frac{\alpha}{27}$, then for

$$N > \max \left\{ \frac{81}{\alpha} - 3, \frac{36}{(1-r)^4} \right\},$$

$F^N * p$ has positive coefficients. Since $\alpha < 1$, we have $r \leq \frac{1}{18}$ and so the second bound can be replaced by 45.3. In particular, it is bounded as $\alpha \rightarrow 0$ and we have that N is asymptotically $\frac{81}{\alpha}$.

Finally, we compute N directly. We claim that for $N \geq \frac{18}{\alpha} - 3$, $(x + y + z)^N p_\alpha$ has non-negative coefficients and that this bound is sharp when $\alpha = 6w^{-1}$ for some integer w .

For $a + b + c = N + 3$, the coefficient of $\frac{N!}{a!b!c!} x^a y^b z^c$ in $(x + y + z)^N p_\alpha$ is easily seen to be $fg - 3h - 2g - (6 - \alpha)f$, where $f = a + b + c$, $g = ab + ac + bc$, and $h = abc$. But this equals

$$(f - 2)g - (9 - \alpha)h = (N + 1)g - (9 - \alpha)h = h \left((N + 1) \frac{g}{h} - (9 - \alpha) \right).$$

Now, $g/h = 1/a + 1/b + 1/c$, and it's easy to show that if $a, b, c \geq 0$ and $a + b + c = N + 3$, then the minimum occurs when $a = b = c = \frac{N+3}{3}$. That is, $g/h \geq 9/(N + 3)$ and equality holds if 3 divides N . We have

$$\frac{9(N + 1)}{N + 3} - (9 - \alpha) = \alpha - \frac{18}{N + 3},$$

and thus if $N \geq \frac{18}{\alpha} - 3$, all coefficients are non-negative. If $\frac{18}{\alpha} - 3$ is a multiple of 3, i.e., if $w = \frac{6}{\alpha} - 1$ is an integer, then for $N = 3w - 3$ the coefficient of $x^w y^w z^w$ will be 0, hence in this case N is best possible.

As $\alpha \rightarrow 0$, the bound from the theorem will be $\frac{81}{\alpha} - 3$ which has the same order of growth as the true bound $\frac{18}{\alpha} - 3$.

Remark 3. The technique used for forms with zeros at the corners should extend to forms in $Po(n, d)$ with zeros on the boundary and yield a quantitative Pólya's Theorem in this case. Also, using Schweighofer's construction, this should have applications to representations of polynomials nonnegative on compact sets and to optimization. These topics will be the subject of future work.

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