Book Review

From Zero to Infinity (50th Anniversary Edition)

Reviewed by Bruce Reznick

From Zero to Infinity (50th Anniversary Edition)
Constance Reid
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In the interest of full disclosure, readers should know that this was the reviewer's favorite childhood book. As an adult, he had the uncommonly satisfying experience of serving on the MAA Publications Committee when the book was brought back into print in its fourth edition in 1991 after Constance Reid recovered the copyright from its original publisher. The reviewer has had the pleasure of several conversations with the author and was solicited to write a blurb for this edition in advance of its publication. He is not unbiased.

The story of how *From Zero to Infinity* came to be written is close to legendary and serves as the "Author's Note" for the present edition. Here is an excerpt:

> It begins with a phone call from my sister, Julia Robinson, on the morning of January 31, 1952. ... Julia tells me that a program by her husband, Raphael Robinson, had turned up the first new "perfect numbers" in seventy-five years—not one but two of them. ... Julia explains the problem simply: *perfect numbers*—the name itself is intriguing—are numbers like 6 that are the sum of all the divisors except themselves: $6 = 1 + 2 + 3$. Then she tells me there is a particular form of prime necessary for the formation of such numbers, the amount of calculation involved in determining their primality, the enormousness of such primes. For me the whole thing is fascinating. I decide to write an article on the discovery of new perfect numbers. ... It is Emma [Lehmer] who suggests that I send my article to *Scientific American*. ... After reading my article, [publisher Robert L.] Crowell immediately wrote to ask if I would be interested in writing a book on numbers that he could pair with a book on the letters of the alphabet. Even I found the combination a bit incongruous, but it gave me an idea. The title of Mr. Crowell's book, already in print, was *Twenty-six Letters*. I would write a book about the ten digits; and because I had found what Julia had told me to be so interesting, I would call it *What Makes Numbers Interesting*. [pp. xiii–xiv][1]

The article on perfect numbers became the chapter on "6" in the original, 1955 edition. The chapter on "0" mainly discussed positional notation, "1" covered factorization and primes, "2" was binary

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[1] *All pagination refers to the 2005 edition under review.*

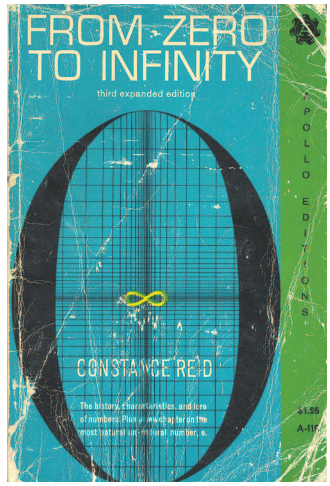
arithmetic, "3" more on primes and prime testing, "4" talked about squares and the Pythagorean theorem, "5" was on pentagonal numbers and Euler's generating function for the partition function, "7" discussed constructible polygons and Fermat numbers, "8" touched on Waring's Problem, and "9" introduced congruences and quadratic reciprocity. There is a short paragraph on "..." giving the well-known proof by contradiction that there are no uninteresting numbers. Each chapter concludes with a short set of questions, and this was part of the book that influenced me most.

> The book was finished in a little over a year, then came a problem. The sales department flatly vetoed my proposed title—*What Makes Numbers Interesting*. The word *interesting* bothered them. Nobody would buy a book about things that were described as "interesting".... The sales department simply loved the one that I disliked the most—*From Zero to Infinity*. My reasons for disliking it were the following. First, in ascending order of importance, it was similar to the title of a then very popular novel, *From Here to Eternity*. ... Second, it was too similar to George Gamow's *One, Two, Three... Infinity* (although Gamow had begun with the number 1 while I had begun with the number 0). My real objection to the proposed title, however, was that I had not written anything in my book about the theory of the infinite. [p. xv]

As a possible compromise, the publisher's invoice on the reviewer's copy referred to *From Hero to Infinity*. A comparison of this edition to the original shows that there are now an "Author's Note" and two additional chapters added back in the 1960s: one on "Euler's constant" (oddly, not placed between "2" and "3") and one on "Aleph-Zero".

The email message from the *Notices* asking me to review this book noted that the book is already well known among mathematicians, so rather than giving a detailed description of the book, the review should serve as a "jumping-off point" for an essay about the book's contents. "Such an essay might comment on how the book influenced you personally..." the email said.

Before I leap into a reverie of possibly unreliable childhood memories, I want to say that, upon reading this edition anew, I was struck by its superb mathematical taste. The author knows what is important, what to talk about, and what to omit.



Cover of the third edition. Image by Tori Corkery.

I was the sort of child who always carried a book wherever he went. In fifth and sixth grades, that book was most frequently *From Zero to Infinity*. Like many readers of this journal, I have always been fascinated by numbers. Some time before my fifth birthday, my parents remember hearing me announce that it was "July 48". No, they corrected me, it's "August 17". "But they're the same thing," I am alleged to have replied. My father was writing at the time for the Robert Q. Lewis comedy variety show on radio, and he would bring home the unused tickets so I could play with the numbers printed on them. I had favorites (suspiciously, in retrospect): those ending in "1","4","5","6" and "9". Of these, "4" was the very best, but only if it was written as an isoceles right triangle with extended legs. The open version of the numeral with parallel vertical lines was somehow frightening to me then.

The wind was at my back as a budding mathematician: my parents were supportive of any interest that my brother or I might have and could afford to buy me the books I wanted, and my school, Hunter College Elementary School in New York City, was a laboratory for teaching techniques. A kind and gifted high school teacher, Dr. Harry D. Ruderman, visited HCES regularly and met with me periodically to explain the wonders of mathematics, many of which I understood. My parents would go to used magazine stores on Second Avenue and bring back copies of *Scientific American*, at first because they'd have those attached "reader response" cards listing numbers from 1 to 600, but later because of Martin Gardner's column. A birthday treat was to be let loose in a book store with a fixed budget. I'm sure that is where I first saw *From Zero to Infinity*.

I was entranced, even though some of the material was too advanced for me, such as Euler's infinite product. The story of how the new perfect numbers were discovered was as exciting as any childhood adventure story:

> The program had to be written entirely in machine language. One hundred and eighty-four separate commands were necessary to tell the SWAC how to test a possible

> prime by the Lucas method. The same program of commands, however, could be used for testing any number of the Mersenne type from $2^3 - 1$ to $2^{2297} - 1$. The latter was the largest that could be handled. [p. 93]

I had no idea what "machine language" could be, but I was hooked. This was much better than the *Justice League of America*. I knew I was meant to spend my life loving numbers and working with them; *From Zero to Infinity* crucially told me that there was a large community of People of Number I could hope to join when I grew up. (And it was particularly exciting decades later to hear John Selfridge, who searched for Mersenne primes in the 1960s, describe the process. You punched the cards, loaded them into the machine, and started it up. You could tell by the *sound* of the computer's actions when you found a new one!)

To be sure, I had other interests, but they were inextricably wound up with mathematics. I wondered how "6" could be a perfect number if it was on the back of Clete Boyer's Yankees uniform while Mickey Mantle wore a "7".

Nothing in *From Zero to Infinity* grabbed my attention as much as the challenge presented in the appendix to Chapter "4":

> There is nothing to keep a person occupied like trying to represent all numbers by four 4's. All four 4's must be used for every number, but various mathematical notations may also be used, as in the four examples below.

$$1 = \frac{44}{44}, \quad 2 = \frac{4 \times 4}{4 + 4}, \quad 3 = 4 - \left(\frac{4}{4}\right)^4,$$
$$4 = 4 + 4 - \sqrt{4} - \sqrt{4}.$$

> Try now to find similar representations for 5 through 12 in terms of four 4s. [p. 71]

Printed upside down on the same page was one such list of representations, and then the most consequential sentence:

> There is no need to stop with 12, for it is possible, if we do not limit ourselves as to notations, to represent *all* numbers by four 4's. [p. 71]

For this ten-year-old, here is what *From Zero to Infinity* really meant. The game was afoot!

Around the same time, the problem of the four 4's arose as a topic in Martin Gardner's "Mathematical Games" column of January 1964 (quoted here from its reprint in *The Magic Numbers of Dr. Matrix*)[2], and the formulation here allowed more flexibility in the use of notations.

> One seeks to form as many whole numbers as possible, starting with 1, by using only the digit 4 four times—no more, no less—together with simple mathematical symbols. Naturally one must establish which is meant by a "simple" symbol. This traditionally includes the arithmetical signs..., together with the square-root sign (repeated as many finite times as desired), parentheses, decimal points and the factorial sign. ... A decimal point may also be placed above .4, in which case it indicates the repeating decimal .4444... or $\frac{4}{9}$. [p. 49]

In the article, Dr. Matrix traces the history back to 1881, adding

> ... there have been scores of subsequent articles, including tables that go above 2000. Even now the mania will suddenly seize the employees of an office or laboratory, sometimes causing a work stoppage that lasts for days. [p. 50]

The mania lasted for weeks for me, and my work product is preserved on a tightly wound scroll of adding machine tape still buried in a box somewhere in my study. I had to adjudicate my own rules, so that $.\dot{4}$ was okay, but $.\dot{\sqrt{4}}$ wasn't, on the grounds that once you take $\sqrt{4}$, it ceases to be a numeral and becomes an integer. (I didn't think 4!4! should be 2424 either.) I'd earlier scoured the HCES library for math books and was fascinated by the older algebra texts, the sort which computed $\cos(18^o)$ and gave the explicit solutions to the cubic and quartic equations. I'd discovered the somewhat obscure notation $!n$ or *subfactorial*, which counts the number of derangements of $\{1, \ldots, n\}$. If factorial was allowed, then surely subfactorial should be too. (In writing this review, I looked up the 1911 edition of W. W. Rouse Ball's *Mathematical Recreations and Essays* and saw that Rouse Ball takes credit for allowing both factorials in the problems (Errata and Addenda, p. 1).) Conveniently enough, $!4 = 9$, so $\sqrt{!4} = 3$ and $(\sqrt{!4})! = 6$ can be constructed with only one 4.

Every Friday afternoon in school, we were given a large sheet of newsprint paper to draw free-style and express our creativity. Every Friday afternoon, my sheet of newsprint was filled with numbers, as I tuned out the rest of the world and concentrated

[2] *Prometheus Books, Buffalo, NY, 1985.*

my attention on the arithmetic at hand. This feeling of oneness with the subject has, fortunately, not left me as I've grown older.

There has been considerable literature on this topic in the subsequent forty years, easily accessible via your favorite search engine. In an explicit response to one of Dr. Matrix's challenges, a newly minted mathematics Ph.D. named Donald Knuth wrote an article[3] showing how to write 64 using one 4 and lots and lots of square roots, factorials, and greatest integer (not yet floor) functions. He reported that every integer $n < 208$ had such an expression, the limitations being round-off error. Tragically, this paper does not appear in *Mathematical Reviews*, and I do not know the current status of the problem. More recently, a computer scientist named David A. Wheeler, in a webpage[4] called "The definitive four fours answer key", links a 368-page PDF file giving expressions of integers up to 40,000. He presents an elaborate metric for evaluating such expressions. Somewhat strangely, Wheeler allows Γ (which seems to me to be a letter, not punctuation or symbolism) and eschews both kinds of factorial. He also allows "%" to indicate division by 100. (Like most mathematicians, I've never really liked "%" and try not to use it.) For example, the following is taken from p. 29 of his file:

$$2006 = \frac{4+4}{.4\%} + \Gamma(4).$$

I have gone on at such length about this problem because "The Four 4's" is simple to state and impossible to master, and deciding the nature of an acceptable solution is part of the problem. In this way, it was a far more useful introduction to mathematical research than most classroom presentations.

As I got older, I continued to read *From Zero to Infinity* and the other influential math books I knew, among them: Constance Reid's *Introduction to Higher Mathematics* (which was less numerical and which I didn't understand as well), George Gamow's *One, Two, Three... Infinity*, E. T. Bell's *Men of Mathematics*, and all of Martin Gardner's collections. As I moved into high school, I devoured *Recreations in the Theory of Numbers* by A. H. Beiler. Fortunately for today's Young Person of Number, these books are all still in print. The real transition came when my parents asked me what I wanted for my sixteenth birthday. We were living in Los Angeles by then, and I'm sure I was the only kid at Uni High who got L. E. Dickson's three-volume *History of the Theory of Numbers*, which Martin Gardner had cited so often. My classmates' birthday cars are rusted, but the three volumes of Dickson continue to inspire my work.

Accompanying this review is a picture showing the well-worn cover of my oldest copy of *From Zero to Infinity*. Careful readers will note that this is the third edition from 1966. What happened is that my first copy (of the second edition) disintegrated from use; this is the replacement.

I was truly fortunate to have run across *From Zero to Infinity* when I did, and I can only wish Constance Reid and her audience another

$$44 + \frac{4!}{4} = \frac{4! - \sqrt{\sqrt{4^4}}}{.4} = 4! + (\sqrt{4\cdot 4})! + \sqrt{4} = \left\lceil \frac{44}{.\dot{4}\cdot\sqrt{4}} \right\rceil$$

years of great reading.

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[3] *Representing numbers using only one 4*, Mathematics Magazine, ***37*** *1964, pp. 308–310.*

[4] `http://www.dwheeler.com/fourfours/`.