REGULARITY PROPERTIES OF THE STERN ENUMERATION OF THE RATIONALS

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Abstract. The Stern sequence $s(n)$ is defined by $s(0) = 0, s(1) = 1$, $s(2n) = s(n)$, $s(2n+1) = s(n) + s(n+1)$. Stern showed in 1858 that $gcd(s(n), s(n+1)) = 1$, and that for every pair of relatively prime positive integers (a, b) there exists a unique $n \geq 1$ with $s(n) = a$ and $s(n+1) = b$. We show that in a strong sense, the average value of $\frac{s(n)}{s(n+1)}$ is $\frac{3}{2}$, and that for $d \geq 2$, $(s(n), s(n+1))$ is uniformly distributed among all feasible pairs of congruence classes modulo d. More precise results are presented for $d = 2$ and 3.

1. Introduction and History

In 1858, M. A. Stern [18] defined the *diatomic array*, an unjustly neglected mathematical construction. It is a Pascal triangle with memory: each row is created by inserting the sums of pairs of consecutive elements into the previous row.

$$
\begin{array}{lllllllll}
a & b \\
a & a+b & b \\
a & 2a+b & a+b & a+2b & b \\
a & 3a+b & 2a+b & 3a+2b & a+b & 2a+3b & a+2b & a+3b & b \\
 & \vdots
\end{array}
\tag{1.1}
$$

When $(a, b) = (0, 1)$, it is easy to see that each row of the diatomic array repeats as the first half of the next row down. The resulting infinite *Stern sequence* can also be defined recursively by:

$$
s(0) = 0,\ s(1) = 1, \qquad s(2n) = s(n),\ s(2n+1) = s(n) + s(n+1). \tag{1.2}
$$

Taking $(a, b) = (1, 1)$ in (1.1), we obtain blocks of $(s(n))$ for $2^r \leq n \leq 2^{r+1}$. Although $s(2^r) = 1$ is repeated at the ends, each pair $(s(n), s(n+1))$ appears below exactly

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once as a consecutive pair in a row:

$$
\begin{array}{ll}
(r=0) & 1\quad 1 \\
(r=1) & 1\quad 2\quad 1 \\
(r=2) & 1\quad 3\quad 2\quad 3\quad 1 \\
(r=3) & 1\quad 4\quad 3\quad 5\quad 2\quad 5\quad 3\quad 4\quad 1 \\
\vdots &
\end{array}
\tag{1.3}
$$

Mirror symmetry (or an easy induction) implies that for $0 \le k \le 2^r$, we have

$$s(2^r + k) = s(2^{r+1} - k). \tag{1.4}$$

In his original paper, Stern proved that for all n,

$$\gcd(s(n), s(n+1)) = 1; \tag{1.5}$$

moreover, for every pair of positive relatively prime integers (a, b), there is a unique n so that $s(n) = a$ and $s(n+1) = b$. Stern's discovery predates Cantor's proof of the countability of $\mathbb{Q}$ by fifteen years. This property of the Stern sequence has been recently made explicit and discussed in [4]. Another enumeration of the positive rationals involves the *Stern-Brocot array*, which also predates Cantor; see [8], pp. 116–123, 305–306. This was used by Minkowski in defining his ?-function; see [14]. The Stern sequence and Stern-Brocot array make brief appearances in Dickson's *History*, see [6], pp. 156, 426. Apparently, de Rham [5] was the first to consider the sequence $(s(n))$ *per se*, attributing the term "Stern sequence" to Bachmann [2], p. 143, who had only considered the array. The Stern sequence has recently arisen as well in the discussion of 2-regular sequences [1] and the Tower of Hanoi graph [10]. Some other Stern identities and a large bibliography relating to the Stern sequence are given in [19]. A further discussion of the Stern sequence will be found in [16].

Let

$$t(n) = \frac{s(n)}{s(n+1)}. \tag{1.6}$$

Here are blocks of $(t(n))$, for $2^r \le n < 2^{r+1}$ for small r:

$$
\begin{array}{ll}
(r=0) & \frac{1}{1} \\
(r=1) & \frac{1}{2}\quad \frac{2}{1} \\
(r=2) & \frac{1}{3}\quad \frac{3}{2}\quad \frac{2}{3}\quad \frac{3}{1} \\
(r=3) & \frac{1}{4}\quad \frac{4}{3}\quad \frac{3}{5}\quad \frac{5}{2}\quad \frac{2}{5}\quad \frac{5}{3}\quad \frac{3}{4}\quad \frac{4}{1} \\
\vdots &
\end{array}
\tag{1.7}
$$

In Section 3, we shall show that

$$\sum_{n=0}^{N-1} t(n) = \frac{3N}{2} + \mathcal{O}(\log^2 N), \tag{1.8}$$

so the "average" element in the Stern enumeration of $\mathbb{Q}_+$ is $\frac{3}{2}$.

For a fixed integer $d \geq 2$, let

$$S_d(n) := (s(n) \bmod d, s(n+1) \bmod d) \tag{1.9}$$

and let

$$\mathcal{S}_d = \{(i \bmod d, j \bmod d) : \gcd(i,j,d) = 1\}. \tag{1.10}$$

It follows from (1.5) that $S_d(n) \in \mathcal{S}_d$ for all n. In Section 4, we shall show that for each d, the sequence $(S_d(n))$ is uniformly distributed on $\mathcal{S}_d$, so the "probability" that $s(n) \equiv i \pmod d$ can be explicitly computed. More precisely, let

$$T(N;d,i) = \left|\{n : 0 \leq n < N \ \& \ s(n) \equiv i \bmod d\}\right|. \tag{1.11}$$

Then there exists $\tau_d < 1$ so that

$$T(N;d,i) = r_{d,i}N + \mathcal{O}(N^{\tau_d}), \tag{1.12}$$

where

$$r_{d,i} = \frac{1}{d} \cdot \prod_{p|i,p|d} \frac{p}{p+1} \cdot \prod_{p\nmid i,p|d} \frac{p^2}{p^2-1}. \tag{1.13}$$

In particular, the probability that $s(n)$ is a multiple of d is $I(d)^{-1}$, where

$$I(d) = d\prod_{p\ |\ d} \frac{p+1}{p} \in \mathbb{N}. \tag{1.14}$$

In Section 5, we present more specific information for the cases $d = 2$ and 3. It is an easy induction that $s(n)$ is even if and only if n is a multiple of 3, so that $\tau_2 = 0$. We show that $\tau_3 = \frac{1}{2}$ and give an explicit formula for $T(2^r; 3, 0)$, as well as a recursive description of those n for which $3 \mid s(n)$. We also prove that, for all $N \geq 1$, $T(N;3,1) - T(N;3,2) \in \{0,1,2,3\}$.

It will be proved in [16] that

$$T(2^r;4,0) = T(2^r;5,0), \qquad T(2^r;6,0) = T(2^r;9,0) = T(2^r;11,0); \tag{1.15}$$

we conjecture that $T(2^r;22,0) = T(2^r;27,0)$. (The latter is true for $r \leq 19$.) These exhaust the possibilities for $T(2^r;N_1,0) = T(2^r;N_2,0)$ with $N_i \leq 128$. Note that $I(4) = I(5) = 6$, $I(6) = I(8) = I(9) = I(11) = 12$ and $I(22) = I(27) = 36$. However, $T(2^r;8,0) \neq T(2^r;6,0)$, so there is more than just asymptotics at work.

2. Basic facts about the Stern sequence

We formalize the definition of the diatomic array. Define $Z(r,k) = Z(r,k;a,b)$ recursively for $r \ge 0$ and $0 \le k \le 2^r$ by:

$$Z(0,0) = a, \quad Z(0,1) = b; \tag{2.1}$$
$$Z(r+1,2k) = Z(r,k), \quad Z(r+1,2k+1) = Z(r,k) + Z(r,k+1).$$

The following lemma follows from (1.2), (2.1) and a simple induction.

Lemma 2.1. *For $0 \le k \le 2^r$, we have*

$$Z(r,k;0,1) = s(k). \tag{2.2}$$

Lemma 2.1 leads directly to a general formula for the diatomic array.

Theorem 2.2. *For $0 \le k \le 2^r$, we have*

$$Z(r,k;a,b) = s(2^r - k)a + s(k)b. \tag{2.3}$$

Proof. Clearly, $Z(r,k;a,b)$ is linear in (a,b) and it also satisfies a mirror symmetry

$$Z(r,k;a,b) = Z(r,2^r - k;b,a) \tag{2.4}$$

for $0 \le k \le 2^r$, c.f. (1.4). Thus,

$$Z(r,k;a,b) = aZ(r,k;1,0) + bZ(r,k;0,1) = aZ(r,2^r - k;0,1) + bZ(r,k;0,1). \tag{2.5}$$

The result then follows from Lemma 2.1. □

The diatomic array contains a self-similarity: any two consecutive entries in any row determine the corresponding portion of the succeeding rows. More precisely, we have a relation whose simple inductive proof is omitted, and which immediately leads to the iterated generalization of (1.2).

Lemma 2.3. *If $0 \le k \le 2^r$ and $0 \le k_0 \le 2^{r_0} - 1$, then*

$$Z(r + r_0, 2^r k_0 + k; a, b) = Z(r,k; Z(r_0,k_0;a,b), Z(r_0,k_0+1;a,b)). \tag{2.6}$$

Corollary 2.4. *If $n \ge 0$ and $0 \le k \le 2^r$, then*

$$s(2^r n + k) = s(2^r - k)s(n) + s(k)s(n+1). \tag{2.7}$$

Proof. Take $(a,b,k_0,r_0) = (0,1,n,\lceil \log_2(n+1) \rceil)$ in Lemma 2.3, so that $k_0 < 2^{r_0}$, and then apply Theorem 2.2. □

We turn now to $t(n)$. Clearly, $t(2n) < 1 \le t(2n+1)$ for all n; after a little algebra, (1.2) implies

$$t(2n) = \cfrac{1}{1 + \cfrac{1}{t(n)}}, \qquad t(2n+1) = 1 + t(n). \tag{2.8}$$

The mirror symmetry (1.4) yields two other formulas which are evident in (1.7):

$$t(2^r + k)t(2^{r+1} - k - 1) = 1, \tag{2.9}$$

for $0 \le k \le 2^r - 1$, which follows from

$$t(2^{r+1} - k - 1) = \frac{s(2^{r+1} - k - 1)}{s(2^{r+1} - k)} = \frac{s(2^r + k + 1)}{s(2^r + k)} = \frac{1}{t(2^r + k)}; \tag{2.10}$$

and

$$t(2^r + 2\ell) + t(2^{r+1} - 2\ell - 2) = 1, \tag{2.11}$$

for $r \ge 1$ and $0 \le 2\ell \le 2^r - 2$, which follows from

$$\frac{s(2^r + 2\ell)}{s(2^r + 2\ell + 1)} + \frac{s(2^{r+1} - 2\ell - 2)}{s(2^{r+1} - 2\ell - 1)} = \frac{s(2^r + 2\ell)}{s(2^r + 2\ell + 1)} + \frac{s(2^r + 2\ell + 2)}{s(2^r + 2\ell + 1)}, \tag{2.12}$$

since $s(2m) + s(2m + 2) = s(2m + 1)$.

Although we will not use it directly here, we mention a simple closed formula for $t(n)$, and hence for $s(n)$. Stern had already proved that if $2^r \le n < 2^{r+1}$, then the sum of the denominators in the continued fraction representation of $t(n)$ is $r+1$; this is clear from (2.8). Lehmer [11] gave an exact formulation, of which the following is a variation. Suppose n is odd and $[n]_2$, the binary representation of n, consists of a block of a_1 1's, followed by a_2 0's, a_3 1's, etc, ending with a_{2v} 0's and a_{2v+1} 1's, with $a_j \ge 1$. (That is, $n = 2^{a_1+\cdots+a_{2v+1}} - 2^{a_2+\cdots+a_{2v+1}} \pm \cdots \pm 2^{a_{2v+1}} - 1$.) Then

$$t(n) = \frac{s(n)}{s(n+1)} = \frac{p}{q} = a_{2v+1} + \cfrac{1}{a_{2v} + \cfrac{1}{\cdots + \cfrac{1}{a_1}}}. \tag{2.13}$$

Conversely, if $\frac{p}{q} > 1$ and (2.13) gives its presentation as a simple continued fraction with an odd number of denominators, then the unique n with $t(n) = \frac{p}{q}$ has the binary representation described above. (If n is even or $\frac{p}{q} < 1$, apply (2.9) first.)

The *Stern-Brocot array* is named after the clockmaker Achille Brocot, who used it [3] in 1861 as the basis of a gear table; see also [9]. This array caught the attention of several French number theorists, and is discussed in [12]. It is formed by applying the diatomic rule to numerators and denominators simultaneously:

$$\begin{array}{ll} (r=0) & \frac{0}{1}\ \ \frac{1}{0} \\ (r=1) & \frac{0}{1}\ \ \frac{1}{1}\ \ \frac{1}{0} \\ (r=2) & \frac{0}{1}\ \ \frac{1}{2}\ \ \frac{1}{1}\ \ \frac{2}{1}\ \ \frac{1}{0} \\ (r=3) & \frac{0}{1}\ \ \frac{1}{3}\ \ \frac{1}{2}\ \ \frac{2}{3}\ \ \frac{1}{1}\ \ \frac{3}{2}\ \ \frac{2}{1}\ \ \frac{3}{1}\ \ \frac{1}{0} \\ \vdots & \end{array} \tag{2.14}$$

This array is not quite the same as (1.7). If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive in the r-th row, then they repeat in the $(r+1)$-st row, separated by $\frac{a+c}{b+d}$. It is easy to see that the elements of the r-th row are $\frac{s(k)}{s(2^r-k)}$, $0 \le k \le 2^r$. It is also easy to show that the elements of each row are increasing, and moreover, that they share a property with the Farey sequence.

Lemma 2.5. *For* $0 \le k \le 2^r - 2$*,*

$$\frac{s(k+1)}{s(2^r-k-1)} - \frac{s(k)}{s(2^r-k)} = \frac{1}{s(2^r-k)s(2^r-k-1)}. \tag{2.15}$$

That is,

$$s(k+1)s(2^r-k) - s(k)s(2^r-k-1) = 1. \tag{2.16}$$

This lemma has a simple proof by induction, which can be found in [12], p.467 and [8], p.117.

The "new" entries in the $(r+1)$-st row of (2.14) are a permutation of the r-th row of (1.7). The easiest way to express the connection (see [16]) for rationals $\frac{p}{q} > 1$ is that if $0 < k < 2^r$ is odd, then

$$\frac{p}{q} = \frac{s(2^r+k)}{s(2^r-k)} = \frac{s(\overleftarrow{2^r+k})}{s(\overleftarrow{2^r+k}+1)}, \tag{2.17}$$

where $\overleftarrow{n}$ denotes the integer so that $[n]_2$ and $[\overleftarrow{n}]_2$ are the reverse of each other. If $\frac{p}{q} < 1$, then apply mirror symmetry to the instance of (2.17) which holds for $\frac{q}{p}$.

The Minkowski ?-function can be defined using the first half of the rows of (2.14). For odd ℓ, $0 \le \ell \le 2^r$,

$$?\left(\frac{s(\ell)}{s(2^{r+1}-\ell)}\right) = \frac{\ell}{2^r}. \tag{2.18}$$

This gives a strictly increasing map from $\mathbb{Q} \cap [0,1]$ to the dyadic rationals in $[0,1]$, which extends to a continuous strictly increasing map from $[0,1]$ to itself, taking quadratic irrationals to non-dyadic rationals.

Finally, suppose N is a positive integer, written as

$$N = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_v}, \qquad r_1 > r_2 > \cdots > r_v. \tag{2.19}$$

We shall define

$$N_0 = 0; \quad N_j = 2^{r_1} + \cdots + 2^{r_j} \text{ for } j = 1, \ldots, v. \tag{2.20}$$

Further, for $1 \le j \le v$, let $M_j = 2^{-r_j} N_{j+1}$, so that

$$N_j = N_{j-1} + 2^{r_j} = 2^{r_j}(M_j + 1) = 2^{r_{j-1}} M_{j-1}. \tag{2.21}$$

and, for $a < b \in \mathbb{Z}$, let

$$[a,b) := \{k \in \mathbb{Z} : a \le k < b\}. \tag{2.22}$$

Our proofs will rely on the observation that

$$[0,N) = \bigcup_{j=0}^{v-1} [N_j, N_{j+1}) = \bigcup_{j=1}^{v} [2^{r_j} M_j, 2^{r_j}(M_j+1)), \tag{2.23}$$

where the above unions are disjoint, so that, formally,

$$\sum_{n=0}^{N-1} = \sum_{j=0}^{v-1} \sum_{n=N_j}^{N_{j+1}-1} = \sum_{j=1}^{v} \sum_{n=2^{r_j} M_j}^{2^{r_j}(M_j+1)-1} . \tag{2.24}$$

3. The Stern-Average Rational

We begin by looking at the sum of $t(n)$ along the rows of (1.7). Let

$$A(r) = \sum_{n=2^r}^{2^{r+1}-1} t(n) \qquad \text{and} \qquad \tilde{A}(r) = \sum_{n=0}^{2^r-1} t(n) = \sum_{i=0}^{r-1} A(i). \tag{3.1}$$

Lemma 3.1. *For $r \geq 0$,*

$$A(r) = \frac{3}{2} \cdot 2^r - \frac{1}{2} \qquad \textit{and} \qquad \tilde{A}(r) = \frac{3}{2} \cdot 2^r - \frac{r+3}{2}. \tag{3.2}$$

Proof. First note that $A(0) = t(1) = \frac{1}{1} = \frac{3}{2} - \frac{1}{2}$. Now observe that for $r \geq 0$,

$$A(r+1) = \sum_{j=0}^{2^{r+1}-1} t(2^{r+1}+j) = \sum_{k=0}^{2^r-1} t(2^{r+1}+2k) + \sum_{k=0}^{2^r-1} t(2^{r+1}+2k+1). \tag{3.3}$$

Using (2.11) and (2.8), we can simplify this summation:

$$\sum_{k=0}^{2^r-1} t(2^{r+1}+2k) = \frac{1}{2}\left(\sum_{k=0}^{2^r-1} t(2^{r+1}+2k) + t(2^{r+2}-2k-2)\right) = 2^{r-1}, \tag{3.4}$$

and

$$\sum_{k=0}^{2^r-1} t(2^{r+1}+2k+1) = \sum_{k=0}^{2^r-1} \big(1 + t(2^r+k)\big) = 2^r + A(r). \tag{3.5}$$

Thus, $A(r+1) = 2^{r-1} + 2^r + A(r)$, and the formula for $A(r)$ is established by induction. This also immediately implies the formula for $\tilde{A}(r)$. □

Lemma 3.2. *If m is even, then*

$$\tilde{A}(r) \leq \sum_{k=0}^{2^r-1} t(2^r m + k) < A(r). \tag{3.6}$$

Proof. For fixed (k, r), let

$$\Phi_{k,r}(x) = \frac{s(2^r-k)x + s(k)}{s(2^r-(k+1))x + s(k+1)}. \tag{3.7}$$

Then it follows from (2.16) that

$$\Phi'_{k,r}(x) = \frac{s(k+1)s(2^r-k) - s(k)s(2^r-k-1)}{(s(2^r-(k+1))x + s(k+1))^2} > 0. \tag{3.8}$$

Using (2.7), we see that

$$
\begin{aligned}
t(2^r m + k) &= \frac{s(2^r m + k)}{s(2^r m + k + 1)} = \frac{s(2^r - k)s(m) + s(k)s(m+1)}{s(2^r - k - 1)s(m) + s(k+1)s(m+1)} \\
&= \Phi_{k,r}\left(\frac{s(m)}{s(m+1)}\right) = \Phi_{k,r}(t(m)).
\end{aligned} \tag{3.9}
$$

Since m is even, $0 \le t(m) < 1$; monotonicity then implies that

$$
t(k) = \Phi_{k,r}(0) \le t(2^r m + k) < \Phi_{r,k}(1) = t(2^r + k). \tag{3.10}
$$

Summing (3.10) on k from 0 to $2^r - 1$ gives (3.6). □

We use these estimates to establish (1.8).

Theorem 3.3. *If $2^r \le N < 2^{r+1}$, then*

$$
\frac{3N}{2} - \frac{r^2 + 7r + 6}{4} \le \sum_{n=0}^{N-1} t(n) < \frac{3N}{2} - \frac{1}{2}. \tag{3.11}
$$

Proof. Recalling (2.24), we apply Lemma 3.2 for each j, with $r = r_j$ and $m = M_j$, so that

$$
\frac{3}{2} \cdot 2^{r_j} - \frac{r_j + 3}{2} \le \sum_{n=N_{j-1}}^{N_j - 1} t(n) < \frac{3}{2} \cdot 2^{r_j} - \frac{1}{2}. \tag{3.12}
$$

After summing on j, we find that

$$
\frac{3N}{2} - \frac{r_1 + \cdots + r_v + 3v}{2} \le \sum_{n=0}^{N-1} t(n) < \frac{3N}{2} - \frac{v}{2}. \tag{3.13}
$$

To obtain (3.11), note that $\sum r_j + 3v \le \frac{r(r+1)}{2} + 3r + 3 = \frac{r^2+7r+6}{2}$. □

Corollary 3.4.

$$
\sum_{n=0}^{N-1} t(n) = \frac{3N}{2} + \mathcal{O}\left(\log^2 N\right). \tag{3.14}
$$

Since $t(2^r - 1) = \frac{r}{1}$, the true error term is at least $\mathcal{O}(\log N)$. Numerical computations using Mathematica suggest that $\log^2 N$ can be replaced by $\log N \log\log N$. It also seems that, at least for small fixed positive integers t,

$$
\alpha_t := \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \frac{s(n)}{s(n+t)} \tag{3.15}
$$

exists. We have seen that $\alpha_1 = \frac{3}{2}$, and if they exist, the evidence suggests that $\alpha_2 \approx 1.262$, $\alpha_3 \approx 1.643$ and $\alpha_4 \approx 1.161$. We are unable to present an explanation for these specific numerical values.

4. Stern Pairs, mod d

We fix $d \geq 2$ with prime factorization $d = \prod p_\ell^{e_\ell}$, $e_\ell \geq 1$, and recall the definitions of $\mathcal{S}_d$ and $S_d(n)$ from (1.9) and (1.10). Let

$$N_d = |\mathcal{S}_d|\,, \tag{4.1}$$

and for $0 \leq i < d$, let

$$N_d(i) = |\{j \bmod d : (i \bmod d, j \bmod d) \in \mathcal{S}_d\}|\,. \tag{4.2}$$

We now give two lemmas whose proofs rely on the Chinese Remainder Theorem.

Lemma 4.1. *The map $S_d : \mathbb{N} \to \mathcal{S}_d$ is surjective.*

Proof. Suppose $\alpha = (i,j) \in \mathcal{S}_d$ with $0 \leq i, j \leq d-1$. We shall show that there exists $w \in \mathbb{N}$ so that $\gcd(i, j + wd) = 1$. Consequently, there exists n with $s(n) = i$ and $s(n+1) = j + wd$, so that $S_d(n) = \alpha$.

Write $i = \prod_\ell q_\ell^{f_\ell}$, $f_\ell \geq 1$, with q_ℓ prime. If $q_\ell \mid j$, then $q_\ell \nmid d$. There exists $w \geq 0$ so that $w \equiv d^{-1} \pmod{q_\ell^{f_\ell}}$ if $q_\ell \mid j$ and $w \equiv 0 \pmod{q_\ell^{f_\ell}}$ if $q_\ell \nmid j$. Then $j + wd \not\equiv 0 \pmod{q_\ell^{f_\ell}}$ for all ℓ, so no prime dividing i divides $j + wd$, as desired. □

Lemma 4.2. *For $0 \leq i \leq d-1$,*

$$N_d = d^2 \prod_\ell \frac{p_\ell^2 - 1}{p_\ell^2} \qquad \textit{and} \qquad N_d(i) = d \prod_{p_\ell \mid i} \frac{p_\ell - 1}{p_\ell}. \tag{4.3}$$

Proof. To compute N_d, we use the Chinese Remainder Theorem by counting the choices for $(i \bmod p_\ell^{e_\ell}, j \bmod p_\ell^{e_\ell})$ for each ℓ. Missing are those (i,j) in which p_ℓ divides both i and j, and so the total number of classes is $(p_\ell^{e_\ell} - p_\ell^{e_\ell - 1})^2$ for each ℓ.

Now fix i. If $p_\ell \mid i$, then $(i,j) \in \mathcal{S}_d$ if and only if $p_\ell \nmid j$; if $p_\ell \nmid i$, then there is no restriction on j. Thus, there are either $p_\ell^{e_\ell} - p_\ell^{e_\ell - 1}$ or $p_\ell^{e_\ell}$ choices for j, respectively. □

Suppose $\alpha = (i,j) \in \mathcal{S}_d$; let $L(\alpha) := (i, i+j)$ and $R(\alpha) = (i+j, j)$, where $i+j$ is reduced mod d if necessary. Then $L(\alpha), R(\alpha) \in \mathcal{S}_d$ and the following lemma is immediate.

Lemma 4.3. *For all n, we have $S_d(2n) = L(S_d(n))$ and $S_d(2n+1) = R(S_d(n))$.*

We now define the directed graph $\mathcal{G}_d$ as follows. The vertices of $\mathcal{G}_d$ are the elements of $\mathcal{S}_d$. The edges of $\mathcal{G}_d$ consist of $(\alpha, L(\alpha))$ and $(\alpha, R(\alpha))$ where $\alpha \in \mathcal{S}_d$. Iterating, we see that $L^k(\alpha) = (i, i+kj)$ and $R^k(\alpha) = (i+kj, j)$, so that $L^d = R^d = id$, and $L^{-1} = L^{d-1}$ and $R^{-1} = R^{d-1}$. Thus, if (α, β) is an edge of $\mathcal{G}_d$, then there is a walk of length $d-1$ from β to α.

Each vertex of $\mathcal{G}_d$ has out-degree two; since $(L^{-1}(\alpha), \alpha)$ and $(R^{-1}(\alpha), \alpha)$ are edges, each vertex has in-degree two as well. Let $M_d = [m_{\alpha(d)\beta(d)}] = [m_{\alpha\beta}]$ denote the adjacency matrix for $\mathcal{G}_d$: M_d is the $N_d \times N_d$ 0-1 matrix so that $m_{\alpha L(\alpha)} = m_{\alpha R(\alpha)} = 1$, with other entries equal to 0. For a positive integer r, write

$$M_d^r = [m_{\alpha\beta}^{(r)}]; \tag{4.4}$$

then $m_{\alpha\beta}^{(r)}$ is the number of walks of length r from α to β. Finally, for $\gamma \in \mathcal{S}_d$, and integers $U_1 < U_2$, let

$$B(\gamma; U_1, U_2) = |\{m : U_1 \le m < U_2 \ \&\ S_d(m) = \gamma\}| \tag{4.5}$$

The following is essentially equivalent to Lemma 2.3.

Lemma 4.4. *Suppose $\alpha = S_d(m)$, $\beta \in \mathcal{S}_d$ and $r \ge 1$. Then $B(\beta; 2^r m, 2^r(m+1)) = m_{\alpha\beta}^{(r)}$ is equal to the number of walks of length r in $\mathcal{G}_d$ from α to β.*

Proof. The walks of length 1 starting from α are $(\alpha, L(\alpha))$ and $(\alpha, R(\alpha))$; that is, $(S_d(n), S_d(2n))$ and $(S_d(n), S_d(2n+1))$. The rest is an easy induction. □

Lemma 4.5. *For sufficiently large N, $m_{\alpha\beta}^{(N)} > 0$ for all α, β.*

Proof. Let $\alpha_0 = (0,1) = S_d(0)$. Note that $L(\alpha_0) = \alpha_0$, hence if there is a walk of length w from α_0 to γ, then there are such walks of every length $\ge w$. By Lemma 4.1, for each $\alpha \in \mathcal{S}_d$, there exists n_α so that $S_d(n_\alpha) = \alpha$. Choose r sufficiently large that $n_\alpha < 2^r$ for all α. Then by Lemma 4.4, for every γ, there is a walk of length r from α_0 to γ, and so there is a walk of length $(d-1)r$ from γ to α_0. Thus, for any $\alpha, \beta \in \mathcal{S}_d$, there is at least one walk of length dr from α to β via α_0. □

We need a version of Perron-Frobenius. Observe that $A_d = \frac{1}{2} M_d$ is doubly stochastic and the entries of $A_d^N = 2^{-N} M_d^N$ are positive for sufficiently large N. Thus A_d is *irreducible* (see [13], Ch.1), so it has a simple eigenvalue of 1, and all its other eigenvalues are inside the unit disk. It follows that M_d has a simple eigenvalue of 2. Let

$$f_d(T) = T^k + c_{k-1}T^{k-1} + \cdots + c_0 \tag{4.6}$$

be the minimal polynomial of M_d. Let $\rho_d < 2$ be the maximum modulus of any non-2 root of f_d, and let $1 + \sigma_d$ be the maximum multiplicity of any such maximal root. Then for $r \ge 0$ and all (α, β),

$$m_{\alpha\beta}^{r+k} + c_{k-1} m_{\alpha\beta}^{r+k-1} + \cdots + c_0 m_{\alpha\beta}^r = 0. \tag{4.7}$$

It follows from the standard theory of linear recurrences that for some constants $c_{\alpha\beta}$,

$$m_{\alpha\beta}^r = c_{\alpha\beta} 2^r + (r^{\sigma_d} \rho_d^r) \qquad \text{as } r \to \infty. \tag{4.8}$$

In particular, $\lim_{r\to\infty} A_d^r = A_{d0} := [c_{\alpha\beta}]$, and since $A_d^{r+1} = A_d A_d^r$, it follows that each column of A_{d0} is an eigenvector of A_d, corresponding to $\lambda = 1$. Such eigenvectors are constant vectors and since A_{d0} is doubly stochastic, we may conclude that for all (α, β), $c_{\alpha\beta} = \frac{1}{N_d}$. Then there exists $c_d > 0$ so that for $r \ge 0$ and all (α, β),

$$\left| m_{\alpha\beta}^r - \frac{2^r}{N_d} \right| < c_d r_d^{\sigma_d} \rho_d^r. \tag{4.9}$$

Computations show that for for small values of d at least, $\rho_d = \frac{1}{2}$ and $\sigma_d = 0$. In any event, by choosing $2 > \bar{\rho}_d > \rho_d$ if $\sigma_d > 0$, we can replace $r_d^{\sigma_d}\rho_d^r$ by $\bar{\rho}_d^r$ in the upper bound. Putting this together, we have proved the following theorem.

Theorem 4.6. *There exist constants c_d and $\bar{\rho}_d < 2$ so that if $m \in \mathbb{N}$ and $\alpha \in \mathcal{S}_d$, then for all $r \geq 0$,*

$$\left| B(\alpha; 2^r m, 2^r(m+1)) - \frac{2^r}{N_d} \right| < c_d \bar{\rho}_d^r. \tag{4.10}$$

We now use this result on blocks of length 2^r to get our main theorem.

Theorem 4.7. *For fixed $d \geq 2$, there exists $\tau_d < 1$ so that, for all $\alpha \in \mathcal{S}_d$,*

$$B(\alpha; 0, N) = \frac{N}{N_d} + \mathcal{O}(N^{\tau_d}). \tag{4.11}$$

Proof. By (2.25), we have

$$B(\alpha; 0, N) = \sum_{j=0}^{v-1} B(\alpha; N_j, N_{j+1}) = \sum_{j=1}^{v} B(\alpha; 2^{r_j} M_j, 2^{r_j}(M_j + 1)). \tag{4.12}$$

It follows that

$$\left| B(\alpha; 0, N) - \frac{N}{N_d} \right| \leq c_d(\bar{\rho}_d^{r_1} + \cdots + \bar{\rho}_d^{r_v}). \tag{4.13}$$

If $\bar{\rho}_d \leq 1$, the upper bound is $\mathcal{O}(r_1) = \mathcal{O}(\log N) = \mathcal{O}(N^\epsilon)$ for any $\epsilon > 0$. If $1 \leq \bar{\rho}_d < 2$, the upper bound is $\mathcal{O}(\bar{\rho}_d^{r_1}) = \mathcal{O}(N^{\tau_d})$ for $\tau_d = \frac{\log \bar{\rho}_d}{\log 2}$, since $N \leq 2^{r_1+1}$. □

Using the notation (1.11), we have

$$T(N; d, i) = \sum_{\alpha=(i,j)\in\mathcal{S}_d} B(\alpha; 0, N), \tag{4.14}$$

and the following is an immediate consequence of Lemma 4.2 and Theorem 4.7.

Corollary 4.8. *Suppose $d \geq 2$. Then*

$$T(N; d, i) = r_{d,i} N + \mathcal{O}(N^{\tau_d}), \tag{4.15}$$

where, recalling that $d = \prod p_\ell^{e_\ell}$,

$$r_{d,i} = \frac{1}{d} \cdot \prod_{p_\ell | i} \frac{p_\ell}{p_\ell + 1} \cdot \prod_{p_\ell \nmid i} \frac{p_\ell^2}{p_\ell^2 - 1}. \tag{4.16}$$

For example, if p is prime, then $f(p, 0) = \frac{1}{p+1}$ and $f(p, i) = \frac{p}{p^2-1}$ when $p \nmid i$.

In some sense, the model here is a Markov Chain, if we imagine going from m to $2m$ or $2m+1$ with equal probability, so that the $B(\beta; 2^r m, 2^r(m+1))$'s represent the distribution of destinations after r steps. Ken Stolarsky has pointed out that [17] is a somewhat different application of the limiting theory of Markov Chains in a number theoretic setting.

5. Small values of d

It is immediate to see (and to prove) that $2 \mid s(n)$ if and only if $3 \mid n$, thus $S_2(n)$ cycles among $\{(0,1),(1,1),(1,0)\}$ and $\tau_2 = 0$. This generalizes to a family of partition sequences. Suppose $d \geq 2$ is fixed, and let $b(d;n)$ denote the number of ways that n can be written in the form

$$n = \sum_{i \geq 0} \epsilon_i 2^i, \qquad \epsilon_i \in \{0,\dots,d-1\}, \tag{5.1}$$

so that $b(2;n) = 1$. It is shown in [15] that

$$\sum_{n=0}^{\infty} s(n) X^n = X \prod_{j=0}^{\infty} \left(1 + X^{2^j} + X^{2^{j+1}}\right). \tag{5.2}$$

A standard partition argument shows that

$$\sum_{n=0}^{\infty} b(d;n) X^n = \prod_{j=0}^{\infty} \frac{1 - X^{d\cdot 2^j}}{1 - X^{2^j}}. \tag{5.3}$$

Thus, $s(n) = b(3;n-1)$. An examination of the product in (5.3) modulo 2 shows that $b(d;n)$ is odd if and only if $n \equiv 0,1 \bmod d$ (see [15], Theorems 5.2 and 2.14.)

Suppose now that $d = 3$. Write the 8 elements of $\mathcal{S}_3$ in lexicographic order:

$$(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2). \tag{5.4}$$

Then in the notation of the last section,

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}. \tag{5.5}$$

The minimal polynomial of M_3 is

$$f_3(T) = T^5 - 2T^4 + T^3 - 4T^2 + 4T = T(T-1)(T-2)(T-\mu)(T-\bar{\mu}), \tag{5.6}$$

where

$$\mu = \frac{-1+\sqrt{7}i}{2}, \qquad \bar{\mu} = \frac{-1-\sqrt{7}i}{2}. \tag{5.7}$$

Since the roots of f_3 are distinct, we see that for each $(\alpha,\beta) \in \mathcal{S}_3$, for $r \geq 1$, there exist constants $v_{\alpha\beta i}$ so that

$$m_{\alpha\beta}^{(r)} = v_{\alpha\beta 1} + v_{\alpha\beta 2}\mu^r + v_{\alpha\beta 3}\bar{\mu}^r + \frac{1}{8}\cdot 2^r = \frac{1}{8}\cdot 2^r + \mathcal{O}(2^{r/2}). \tag{5.8}$$

(As it happens, there are only eight distinct sequences $m^{(r)}_{\alpha\beta}$.) Corollary 4.8 then implies that

$$\begin{gathered} T(N;3,0) = \frac{N}{4} + \mathcal{O}(\sqrt{N}), \\ T(N;3,1) = \frac{3N}{8} + \mathcal{O}(\sqrt{N}),\ T(N;3,2) = \frac{3N}{8} + \mathcal{O}(\sqrt{N}). \end{gathered} \tag{5.9}$$

Since $T(N;3,0) + T(N;3,1) + T(N;3,2) = N$, we gain complete information from studying $T(N;3,0)$ and

$$\Delta(N) = \Delta_3(N) := T(N;3,1) - T(N;3,2). \tag{5.10}$$

(That is, $\Delta_3(N+1) - \Delta_3(N)$ equals $0, 1, -1$ when $s(N) \equiv 0, 1, 2 \bmod 3$, respectively.)

To study $T(N;3,0)$, we first define the set $A_3 \subset \mathbb{N}$ recursively by:

$$0, 5, 7 \in A_3, \qquad 0 < n \in A_3 \implies 2n, 8n \pm 5, 8n \pm 7 \in A_3. \tag{5.11}$$

Thus,

$$A_3 = \{0, 5, 7, 10, 14, 20, 28, 33, 35, 40, 45, 47, 49, 51, 56, 61, 63, \dots\}. \tag{5.12}$$

Theorem 5.1. *If $n \ge 0$, then $3 \mid s(n)$ if and only if $n \in A_3$.*

Proof. It follows recursively from (1.2) or directly from (2.7) that

$$s(2n) = s(n), \quad s(8n\pm5) = 2s(n)+3s(n\pm1), \qquad s(8n\pm7) = s(n)+3s(n\pm1). \tag{5.13}$$

Thus, 3 divides $s(n)$ if and only if 3 divides $s(2n), s(8n \pm 5)$ or $s(8n \pm 7)$. Since every $n > 1$ can be written uniquely as $2n', 8n' \pm 5$ or $8n' \pm 7$ with $0 \le n' < n$, the description of A_3 is complete. □

In the late 1970's, E. W. Dijkstra [7](pp. 215–6, 230–232) studied the Stern sequence under the name "fusc", and gave a different description of A_3 (p. 232):

> Inspired by a recent exercise of Don Knuth I tried to characterize the arguments n such that $3 \mid \mathit{fusc}(n)$. With braces used to denote zero or more instances of the enclosed, the vertical bar as the BNF 'or', and the question mark '?' to denote either a 0 or a 1, the syntactical representation for such an argument (in binary) is $\{0\}1\{?0\{1\}0|?1\{0\}1\}?1\{0\}$. I derived this by considering – as a direct derivation of my program – the finite state automaton that computes *fusc* (N) *mod 3.*

Let

$$a_r = |\{n \in A_3 : 2^r \le n < 2^{r+1}\}| = T(2^{r+1};3,0) - T(2^r;3,0). \tag{5.14}$$

It follows from (5.12) that

$$a_0 = a_1 = 0, \quad a_2 = a_3 = a_4 = 2, \quad a_5 = 10. \tag{5.15}$$

Lemma 5.2. *For $r \ge 3$, (a_r) satisfies the recurrence*

$$a_r = a_{r-1} + 4a_{r-3}. \tag{5.16}$$

Proof. This is evidently true for $r = 3, 4, 5$. If $2^r \le n < 2^{r+1}$ and $n = 2n'$, then $2^{r-1} \le n' < 2^r$, so the even elements of A_3 counted in a_r come from elements of A_3 counted in a_{r-1}. If $2^r \le n < 2^{r+1}$ and $n = 8n' \pm 5$ or $n = 8n' \pm 7$, then $2^{r-3} < n' < 2^{r-2}$ and $n' \in A_3$. Thus the odd elements of A_3 counted in a_r come (in fours) from elements of A_3 counted in a_{r-3}. □

The characteristic polynomial of the recurrence (5.16) is $T^3 - T^2 - 4$ (necessarily a factor of $f_3(T)$), and has roots $T = 2,\ \mu$ and $\bar{\mu}$. The details of the following routine computation are omitted.

Theorem 5.3. *For $r \ge 0$, we have the exact formula*

$$a_r = \frac{1}{4} \cdot 2^r + \left(\frac{-7 + 5\sqrt{7}i}{56} \right) \mu^r + \left(\frac{-7 - 5\sqrt{7}i}{56} \right) \bar{\mu}^r. \tag{5.17}$$

Keeping in mind that $s(0) = 0$ is not counted in any a_r, we find after a further computation that the error estimate $\mathcal{O}(\sqrt{N})$ is best possible for $T(N; 3, 0)$:

Corollary 5.4.

$$T(2^r; 3, 0) = \frac{1}{4} \cdot 2^r + \left(\frac{7 - \sqrt{7}i}{56} \right) \mu^r + \left(\frac{7 + \sqrt{7}i}{56} \right) \bar{\mu}^r + \frac{1}{2}\ . \tag{5.18}$$

To study $\Delta(N)$, we first need a somewhat surprising lemma.

Lemma 5.5. *For all N, $\Delta(2N) = \Delta(4N)$.*

Proof. The simplest proof is by induction, and the assertion is trivial for $N = 0$. There are eight possible "short" diatomic arrays modulo 3:

$$\begin{array}{ccccc}
s(N) & & & & s(N+1) \\
s(2N) & & s(2N+1) & & s(2N+2) = \\
s(4N) & s(4N+1) & s(4N+2) & s(4N+3) & s(4N+4)
\end{array}$$

$$\begin{array}{ccccc||ccccc||ccccc||ccccc}
0 & & & & 1 & 0 & & & & 2 & 1 & & & & 0 & 1 & & & & 1 \\
0 & & 1 & & 1 & 0 & & 2 & & 2 & 1 & & 1 & & 0 & 1 & & 2 & & 1 \\
0 & 1 & 1 & 2 & 1 & 0 & 2 & 2 & 1 & 2 & 1 & 2 & 1 & 1 & 0 & 1 & 0 & 2 & 0 & 1 \\
1 & & & & 2 & 2 & & & & 0 & 2 & & & & 1 & 2 & & & & 2 \\
1 & & 0 & & 2 & 2 & & 2 & & 0 & 2 & & 0 & & 1 & 2 & & 1 & & 2 \\
1 & 1 & 0 & 2 & 2 & 2 & 1 & 2 & 2 & 0 & 2 & 2 & 0 & 1 & 1 & 2 & 0 & 1 & 0 & 2
\end{array} \tag{5.19}$$

By counting the elements in the rows mod 3 in each case, we see that $\Delta(2N+2) - \Delta(2N) = \Delta(4N+4) - \Delta(4N)$ is equal to: $1, -1, 2, 0, 1, -2, -1, 0$, respectively. □

Theorem 5.6. *For all n, $\Delta(n) \in \{0,1,2,3\}$. More specifically,*

$$\begin{aligned} S_3(m) = (0,1) &\implies \Delta(2m) = 0,\ \Delta(2m+1) = 0; \\ S_3(m) = (0,2) &\implies \Delta(2m) = 3,\ \Delta(2m+1) = 3; \\ S_3(m) = (1,*) &\implies \Delta(2m) = 1,\ \Delta(2m+1) = 2; \\ S_3(m) = (2,*) &\implies \Delta(2m) = 2,\ \Delta(2m+1) = 1. \end{aligned} \tag{5.20}$$

Proof. To prove the theorem, we first observe that it is correct for $m \le 4$. We now assume it is true for $m \le 2^r$ and prove it for $2^r \le m < 2^{r+1}$. There are sixteen cases: m can be even or odd and there are 8 choices for $S_3(m)$. As a representative example, suppose $S_3(m) = (2,1)$. We shall consider the cases $m = 2t$ and $m = 2t+1$ separately. The proofs for the other seven choices of $S_3(m)$ are very similar and are omitted.

Suppose first that $m = 2t < 2^{r+1}$. Then $S_3(m) = S_3(2t) = (2,1)$, hence $S_3(t) = (2,2)$. We have $\Delta(2m) = 2$ by hypothesis, and hence $\Delta(4m) = 2$ by Lemma 5.5. The eighth array in (5.19) shows that $s(4t) \equiv 2 \bmod 3$, so that $\Delta(4m+1) = \Delta(4m) - 1 = 1$, as asserted in (5.20).

If, on the other hand, $m = 2t+1 < 2^{r+1}$ and $S_3(m) = S_3(2t+1) = (2,1)$, then $S_3(t) = (1,1)$. We now have $\Delta(2t) = 1$ and $\Delta(2t+1) = 2$ by hypothesis and $\Delta(4t) = 1$ by Lemma 5.5. The fourth array in (5.19) shows that $(s(4t), s(4t+1), s(4t+2)) \equiv (1,0,2) \bmod 3$. Thus, it follows that $\Delta(2m) = \Delta(4t+2) = \Delta(4t) + 1 + 0 = 2$ and $\Delta(2m+1) = \Delta(4t+3) = \Delta(4t+2) - 1 = 1$, again as desired. □

Since $S_3(m)$ is uniformly distributed on $\mathcal{S}_3$, (5.20) shows that $\Delta(n)$ takes the values $(0,1,2,3)$ with limiting probability $(\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8})$.

We conclude with a few words about the results announced at the end of the first section, but not proved here. For each (d,i), $T(2^r; d, i)$ will satisfy a recurrence whose characteristic equation is a factor of the minimal polynomial of $\mathcal{S}_d$. It happens that $T(2^r; 4, 0) = T(2^r; 5, 0)$ for small values of r and both satisfy the recurrence with characteristic polynomial $T^4 - 2T^3 + T^2 - 4$ (roots: $2, -1, -\tau, -\bar{\tau}$) so that equality holds for all r. The same applies to $T(2^r; 6, 0) = T(2^r; 9, 0) = T(2^r; 11, 0)$, with a more complicated recurrence. Results similar to Lemma 5.5 and Theorem 5.6 hold for $d = 4$, with a similar proof; Antonios Hondroulis has shown that this is also true for $d = 6$. No result has been found yet for $d = 5$, although a Mathematica check for $N \le 2^{19}$ shows that $-5 \le T(N; 5, 1) - T(N; 5, 4) \le 11$. These topics will be discussed in greater detail in [16].

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