Notes on the equation $x^y = y^x$, Math 296, 2/26/01

These notes will try to organize our discussions on this topic. We began with the observation that $2^4 = 4^2$, and wondered whether this is the only such example. Suppose, more generally, that

$$x^y = y^x, \tag{1}$$

where x and y are positive real numbers. We have a string of equivalent identities:

$$x^y = y^x \implies y \log x = x \ \log y \implies \frac{\log x}{x} = \frac{\log y}{y}. \tag{2}$$

We immediately have to make some restrictions to our conditions on x and y: that we want to exclude the trivial case $x = y$.

It now makes sense to examine the function $f(x) = \frac{\log x}{x}$ for $x > 0$. Basic curve sketching from calculus tells us that $f(x) < 0$ for $0 < x < 1$ and $f(x) > 0$ for $x > 1$ and

$$f'(x) = \frac{x \cdot \frac{1}{x} - \log x \cdot 1}{x^2} = \frac{1 - \log x}{x^2}.$$

Thus, f is increasing on $(0, e)$ and decreasing on (e, ∞). It follows that if $0 < x < 1$, then there is no $y \neq x$ so that (2) holds. A simple exercise in L'Hôpital's Rule shows that $\lim_{x \to \infty} f(x) = 0 = f(0)$. Thus, for every $x \in (1, e)$ there is a unique $y \in (e, \infty)$ for which (2) holds. See below the Mathematica graph of the function:

But how can we determine y, given x? There are two reasonable choices: one is to write $y = x^r$, the other is to write $y = tx$, but as we'll see, these are really the same.

In the first case, if write $y = x^r$, $r \neq 1$, then the soltuions $(2, 4)$ and $(4, 2)$ correspond to $r = 2$ and $r = \frac{1}{2}$, respectively.) With this substitution, into (2), we have

$$\begin{aligned} \frac{\log x}{x} = \frac{\log(x^r)}{x^r} &\implies \frac{\log x}{x} = \frac{r \log x}{x^r} \implies x^{r-1} = r \\ &\implies x = r^{\frac{1}{r-1}} \qquad \text{and} \qquad y = x^r = r^{\frac{r}{r-1}}. \end{aligned} \tag{3}$$

Notice from (3) that, not only is $y = x^r$, but also $y = rx$! This answers the first question about real solutions to (1), but leaves uncertain the questions about integral or rational solutions. Howver, the smaller one of x and y must be in $(1, e)$, and 2 is the only integer in this interval, hence the larger one must be the unique number $v > e$ so that $f(v) = f(2)$, and our analysis shows that v has to be 4.

Are there rational solutions? Well, if x and y are rational, then so is $r = \frac{y}{x}$. We've already tried the simplest values for r; the next one appears to be $r = 3$. But then

$$x = 3^{\frac{1}{2}} = \sqrt{3}, \qquad y = 3\sqrt{3},$$

which are not rational. At this point, it's tempting to note that if we picked r such that $\frac{1}{r-1}$ was an integer, then we'd be guaranteed that x and y are rational. Indeed,

$$\frac{1}{r-1} = m \implies r = 1 + \frac{1}{m} \implies \frac{r}{r-1} = m + 1.$$

In this case, we get a flashback from calculus:

$$(4) \qquad (x_m, y_m) = \left(\left(1 + \frac{1}{m}\right)^m, \left(1 + \frac{1}{m}\right)^{m+1} \right).$$

Recall that $\{x_m\}$ is a sequence that increases to e and $\{y_m\}$ is a sequence that decreases to e. Certainly, $\frac{y_m}{x_m} = 1 + \frac{1}{m} \to 1$ as $m \to \infty$. Putting in some numbers into (4), we find

$$(x_1, y_1) = (2, 4), \qquad (x_2, y_2) = \left(\frac{9}{4}, \frac{27}{8}\right), \qquad (x_3, y_3) = \left(\frac{64}{27}, \frac{256}{81}\right).$$

Are the (x_m, y_m)'s the only rational solutions to (1)? The answer to this question requires some number theory, specifically, the uniqueness of prime factorization of positive integers.

If m and n are positive integers, then we say that m *divides* n (written $m \mid n$), if $\frac{n}{m}$ is an integer; alternately, if there exists an integer d so that $n = md$. A *prime* number is a number p with the property that if $m \mid p$, then $m = 1$ or $m = p$. In other words, the only ways to write p as a product of positive integers are $p = 1 \cdot p$ and $p = p \cdot 1$.

One of the most important theorems in Number Theory is called the **Fundamental Theorem of Arithmetic**. This says that if n is a positive integer, then there exist prime numbers p_j, $p_1 < p_2 \cdots < p_r$, and positive integers e_j so that

$$(5) \qquad n = p_1^{e_1} \cdots p_r^{e_r}.$$

Furthermore, this representation (5) is unique: there is no other way to express n as a product of powers of increasing prime numbers. For this reason, we can say that (5) is **the** *prime factorization* of n. We won't prove this here, but it can be found in many textbooks.

For any prime p, define $\nu_p(n)$ to be the exponent of p in the prime factorization of n. If (5) holds, for example, then $\nu_{p_j}(n) = e_j$ for $1 \le j \le r$ and $\nu_p(n) = 0$ for all primes p which aren't among the p_j's. It is easy to see if you look at it the right way that ν_p is a kind of logarithm: $\nu_p(mm) = \nu_p(m) + \nu_p(n)$. The informal proof is that you count the factors of p that appear in m and n separately, and then in $m \times n$ together.

A fancy way of saying something easy is the following:

$$m \mid n \iff \nu_p(m) \le \nu_p(n) \qquad \text{for every prime } p.$$

Finally, we remark that $\nu_p(n^k) = k\nu_p(n)$ for all positive integers k, and that a converse holds: if m is such that k divides $\nu_p(m)$ for every prime p, then there is an integer t so that $m = t^k$, In fact, if $m = \prod_j p_j^{e_j}$, define $t = \prod_j p_j^{e_j/k}$.

Another definition we will need is this: m and n are *relatively prime* if there are no primes common to the prime factorizations of m and n; that is, if there is no prime p so that $\nu_p(m) > 0$ and $\nu_p(n) > 0$. Put another way, m and n are relatively prime if the fraction $\frac{m}{n}$ is already in lowest terms. For example, $\frac{10}{25} = \frac{2}{5}$, which shows that 10 and 25 are not relatively prime, but 2 and 5 are. If you think about the definition, you'll see that if m and n are relatively prime and s and t are positive integers, then m^s and n^t are also relatively prime. Another necessary fact: Suppose m and n are relatively prime, and m divides the product ns, where s is some integer. Then m divides s. The proof is really simple. What we need to show is that $\nu_p(m) \le \nu_p(s)$ for every prime p. We know that $\nu_p(m) \le \nu_p(ns) = \nu_p(n) + \nu_p(s)$ by hypothesis. If $\nu_p(m) = 0$, then there's nothing to prove, because $\nu_p(s) \ge 0$. And if $\nu_p(m) > 0$, then $\nu_p(n) = 0$ by hypothesis, and there's also nothing to prove!

So now, let's return to the original problem. Suppose $x = r^{\frac{1}{r-1}}$ and $y = r^{\frac{r}{r-1}}$ are both rational numbers. We already know that r is rational. We need to know when x is rational (this would imply that $y = rx$ is rational, too.) Time to write down some letters. Suppose

$$r = \frac{a}{b} \implies \frac{1}{r-1} = \frac{b}{a-b}$$

is rational and larger than 1 (so that $a > b$, and $a - b > 0$). We assume that a and b are relatively prime, so that r has been written in lowest terms. But there's a little more we can say. If b and $a - b$ are *not* relatively prime, then they have a common factor k, so that $b = ks$ and $a - b = kt$ for some integers s and t. But this implies that $a = k(s + t)$, a fact which violates our assumption that a and b are relatively prime. Thus we can also say that b and $a - b$ are relatively prime. Our remaining condition is that x be rational. So let's write that out and set up x as a rational number. We have

$$r^{\frac{1}{r-1}} = \left(\frac{a}{b}\right)^{\frac{b}{a-b}} = \frac{m}{n}. \tag{6}$$

We can assume that m and n are relatively prime. The best thing to do now is to take the $b - a$ powers of both sides in (6), yielding:

$$\frac{a^b}{b^b} = \frac{m^{a-b}}{n^{a-b}} \implies a^b n^{a-b} = b^b m^{a-b}. \tag{7}$$

Let's see what we can get out of (7). We know that a and b are relatively prime, and hence so are a^b and b^b. Since a^b divides $b^b m^{a-b}$ and is relative prime to b^b, it must divide m^{a-b}.

Since m and n (and so m^{a-b} and n^{a-b} are relatively prime, the same reasoning implies that m^{a-b} divides a^b. Thus $a^b = m^{a-b}$, which, together with (7) implies that $b^b = n^{a-b}$.

For any prime p we have

$$b\nu_p(a) = (a-b)\nu_p(m), \tag{8}$$

and since $a-b$ and b are relatively prime, (8) implies that $a - b \mid \nu_p(a)$. Since this is true for every prime p, it follows that there exists an integer u so that

$$a = u^{a-b}. \tag{9}$$

The identical argument, applied to b^b and n^{a-b}, implies that there is an integer v so that

$$b = v^{a-b}. \tag{10}$$

Now, let's write $a = b + k$. (I could have done this much earlier if I'd wanted.) From (9) and (10), the integers b and $b+k$ are both k-th powers of integers. However,

$$k = a - b = u^k - v^k \geq (u+1)^k - u^k \geq 2^k - 1. \tag{11}$$

(The final inequality above follows from the fact that $g(x) = (x+1)^k - x^k$ has a positive derivative for $x > 0$.)

Finally, note that (11) is usually a contradiction! The inequality $k < 2^k - 1$ for $k \geq 2$ is easily established by induction. Thus, (11) implies that $k = 1$. That is, $a = b + 1$ and $r = 1 + \frac{1}{b}$, and upon writing m for b, we get back (4).

It is interesting to play with other, similar equations. For example, suppose

$$x^y = y^{2x}. \tag{12}$$

This time, we do not have any "automatic" solutions. In fact, taking logs and crossdividing gives

$$\frac{\log x}{x} = \frac{2 \log y}{y}. \tag{13}$$

If we put $y = x^r$ into (13) and simplify as before, we get

$$x = (2r)^{\frac{1}{r-1}}, \qquad y = (2r)^{\frac{r}{r-1}}.$$

Notice that $y = 2rx$ this time, and this time, there are at least three values of r which make x and y integers:

$$\begin{aligned} 4^{16} &= 2^{2\cdot 16} \qquad (r = 2) \\ 1^1 &= 1^{2\cdot 1} \qquad (r = \tfrac{1}{2}) \\ 2^{16} &= 16^{2\cdot 2} \qquad (r = 4) \end{aligned}$$

Are these the only ones? (Hint: no!) Can you find them all?

Observe that you can find rational solutions by letting $r = 1 \pm \frac{1}{m}$:

$$x = \left(2 \pm \frac{2}{m}\right)^{\pm m}, \qquad y = \left(2 \pm \frac{2}{m}\right)^{\pm(m+1)}$$

The question of whether these are all the rational solutions is actually a very hard one which has led to a research paper I'm writing with Michael Bennett.