Math 210 Instructor: Ryan Hubscher
Theory of Interest Spring, 2008

Homework Assignment # 10 (max. points = 20)
Due at the beginning of class on Tuesday April 29, 2008

Please show your work - enough to show that you understand how to do the problem. Circle your final answer. Full credit can only be given only if the answer and work leading to the answer are correct.

1. Find the Macaulay duration of a ten-year, $1,000 face value, 8% annual coupon bond. Assume an effective annual interest rate of 7%.

$$\begin{aligned}\bar{d} &= \frac{80v + 2(80v^2) + \cdots + 10(80v^{10}) + 10(1000v^{10})}{80v + 80v^2 + \cdots + 80v^{10} + 1000v^{10}} \\ &= \frac{80(Ia)_{\overline{10|}.07} + 10000v^{10}_{.07}}{80a_{\overline{10|}.07} + 1000v^{10}_{.07}} \\ &= \frac{80(34.7392) + 10000(.508349)}{80(7.02358) + 1000(.508349)} \\ &= \textbf{7.3466 Years}\end{aligned}$$

$$\begin{aligned}v^{10}_{.07} &= 0.508349 \\ a_{\overline{10|}.07} &= \frac{1 - v^{10}}{i} = 7.02358 \\ (Ia)_{\overline{10|}.07} &= \frac{\ddot{a}_{\overline{10|}} - 10v^{10}}{i} \\ &= \frac{(1+i)a_{\overline{10|}} - 10v^{10}}{i} = 34.7391\end{aligned}$$

2. Find the modified duration for the bond in problem (1).

$$\bar{v} = v\bar{d} = \mathbf{6.8660}$$

3. Find the modified duration of a 30-year, $1,000 face value, 6% annual coupon bond. Assume an effective annual interest rate of 9%.

$$\begin{aligned}\bar{d} &= \frac{60v + 2(60v^2) + \cdots + 30(60v^{30}) + 30(1000v^{30})}{60v + 60v^2 + \cdots + 60v^{30} + 1000v^{30}} \\ &= \frac{60(Ia)_{\overline{30|}.09} + 30000v^{30}_{.09}}{60a_{\overline{30|}.09} + 1000v^{30}_{.09}} \\ &= 11.8811 \\ \bar{v} &= v\bar{d} = \mathbf{10.9001}\end{aligned}$$

4. An investment portfolio consists of two bonds: a two-year zero-coupon bond, and a four-year zero-coupon bond. Both bonds have redemption values of $1,000. Assume an effective annual interest rate of 8%. Find the Macaulay duration of the investment portfolio. (The Macaulay duration of a portfolio can be calculated either via all the cash flows of its assets, or as the present-value-weighted-average of the durations of its assets.)

$$\bar{d} = \frac{2(1000v^2) + 4(1000v^4)}{1000v^2 + 1000v^4} = \textbf{2.9232 Years}$$

5. A 10-year $100,000 mortgage will be paid off with level semi-annual amortization payments. Assume that the interest rate is 8%, convertible semi-annually, and that

payments are made at the end of each half-year. Find the Macaulay duration of this mortgage.

$$\begin{aligned}\bar{d} &= \frac{0.5v_{.04} + 1.0v_{.04}^2 + 1.5v_{.04}^3 + \cdots + 10v_{.04}^{20}}{v_{.04} + v_{.04}^2 + v_{.04}^3 + \cdots + v_{.04}^{20}} \\ &= \frac{.5(Ia)_{\overline{20}|.04}}{a_{\overline{20}|.04}} \\ &= \mathbf{4.6046\ Years}\end{aligned}$$

6. You are the actuary for an insurance company. Your company's liabilities include loss reserves, which are liability reserves set aside to make future claim payments on policies which the company has already sold. You believe that these liabilities, totaling $100 million on December 31, 2006, will be paid out according to the following schedule:

Calendar Year	Proportion of Reserves Paid Out
2007	40%
2008	30%
2009	20%
2010	10%

Find the modified duration of your companys loss reserves. Assume that the annual interest rate is 10%, and that all losses paid during a given calendar year are paid at the mid-point of that calendar year.

$$\bar{v} = v\left(\frac{0.5(4v^{0.5}) + 1.5(3v^{1.5}) + 2.5(2v^{2.5}) + 3.5(1v^{3.5})}{4v^{0.5} + 3v^{1.5} + 2v^{2.5} + v^{3.5}}\right) = \mathbf{1.2796}$$

7. Find the modified duration for a share of common stock. Assume that the stock pays annual dividends, with the first dividend of $2 payable 12 months from now, and that subsequent dividends will grow at an annual rate of 4%. Assume that the effective annual interest rate is 9%.

$$P(i) = \frac{D_1}{i-g} = D_1(i-g)^{-1}$$

$$P'(i) = -D_1(i-g)^{-2}$$

$$\bar{v} = -\frac{P'(i)}{P(i)} = \frac{D_1(i-g)^{-2}}{D_1(i-g)^{-1}} = \frac{1}{i-g} = \mathbf{20}$$

8. An insurance company has an obligation to pay 10,000 at the end of each year for 2 years. The insurance company purchases a combination of the following two bonds (both with $1,000 par and redemption values) in order to exactly match its obligation:

Bond A: A 1-year 5% annual coupon bond with a yield rate of 6%.
Bond B: A 2-year 8% annual coupon bond with a yield rate of 7%.

Find the numbers (which need not be integers) of each bond the insurer must purchase to exactly match its obligations.

Time	Bond A	Bond B	Liabilities
1	1050	80	10000
2		1080	10000

$$\begin{aligned} 1050a + 80b &= 10000 \\ 1080b &= 10000 \end{aligned} \quad \Rightarrow \quad \begin{aligned} a &= \mathbf{8.8183\ Bonds} \\ b &= \mathbf{9.2593\ Bonds} \end{aligned}$$

9. Continuing problem (8) above, find the total cost to the insurer of purchasing the needed numbers of bonds.

$$8.8183(1050v_{.06}) + 9.2593(80v_{.07} + 1080v_{.07}^2) = \$\mathbf{18161.82}$$

10. An insurer must pay 2,000 and 1,000 at the ends of years 1 and 2, respectively. The only investments available to the company are a one-year zero-coupon bond (with a par value of 1,000 and an effective annual yield of 9%), and a two-year zero-coupon bond (with a par value of 1,000 and an effective annual yield of 8%). Determine the cost to the company today to match its liabilities exactly.

$$\frac{2000}{1.09} + \frac{1000}{(1.08)^2} = \$\mathbf{2692.20}$$