Math 210 Instructor: Ryan Hubscher
Theory of Interest Spring, 2008

Homework Assignment # 5 (max. points = 20)
Due at the beginning of class on Thursday March 6, 2008

Please show your work - enough to show that you understand how to do the problem. Circle your final answer. Full credit can only be given only if the answer and work leading to the answer are correct.

1. A 15-year annuity-immediate pays 100 the first year, and each subsequent payment is 10% larger then the preceding. Find the present value of this annuity three years before the first payment if the effective annual rate is $i = 5\%$.

   The present value of this annuity one year before the first payment is

$$100\left[\frac{1-\left(\frac{1.1}{1.05}\right)^{15}}{0.05-0.1}\right]$$

   The present value of this annuity three years before the first payment is

$$\left(\frac{1}{1.05}\right)^2 100\left[\frac{1-\left(\frac{1.1}{1.05}\right)^{15}}{0.05-0.1}\right] = \mathbf{1830.98}$$

2. Donna deposits money into an account at the end of every six month period for 5 years. Her deposits follow the following pattern: 500, 500, 500(1.04), 500(1.04), $500(1.04)^2$, $500(1.04)^2, \ldots,$ $500(1.04)^4$, $500(1.04)^4$. If Donna's account earns $i^{(2)} = 6\%$ annual interest, what is the accumulated value of Donna's account 3 years after the last payment.

   A payment of $500(1.04)^{n-1}$ half way into year n and again at the end of year n is equivalent to one payment of

$$500(1.04)^{n-1}s_{\overline{2}|\,.03} = 500(1.04)^{n-1}(1.03+1) = 1015(1.04)^{n-1}$$

   at the end of year n. So this annuity is equivalent to an annuity of 1015 paid at the end of the the first year and another 4 annual payments each 4% greater then the preceding invested at an interest rate $i = (1.03)^2 - 1$ per year. Thus, the accumulated value of this annuity at the time of the last payment is

$$1015\left[\frac{(1.03^2)^5-(1.04)^5}{(1.03)^2-1-.04}\right]$$

   And the accumulated value of this annuity three years after the last payment is

$$1015\left[\frac{(1.03^2)^5-(1.04)^5}{(1.03)^2-1-.04}\right](1.03)^6 = \mathbf{7379.84}$$

3. You are considering the purchase of a share of stock. If you buy the share, you will expect to receive the following dividends: \$3 one year from now, \$3.50 two years from now, \$4 three years from now, and thereafter annual dividends increase by 4% per year, forever. If the effective annual interest rate is 10%, what is the theoretical price of this share of stock?

   Using the *dividend discount model* (Boverman, 110-111) the price of the stock is the present value of all future dividends. The present value (at time 2) of D_3 and all subsequent dividends is $\frac{4}{0.1-0.04}$. This amount must be discounted two more years to find its present value at time zero. Thus, the price of this stock, or the present value of all future dividends is

$$3v + 3.5v^2 + \left(\frac{4}{0.1-0.04}\right)v^2 = \mathbf{60.72}$$

4. A perpetuity-immediate pays \$1,000 the first year, and the annual payment increases by \$100 each year thereafter. If the present value of the perpetuity is \$9,000, what is the annual effective rate of interest?

Time	0	1	2	3	4	⋯
Level Perpetuity		900	900	900	900	⋯
Increasing Perpetuity		100	200	300	400	⋯

$$9000 = 900a_{\overline{\infty}|} + 100(Ia)_{\overline{\infty}|}$$
$$90 = \frac{9}{i} + \frac{1}{i^2} + \frac{1}{i}$$
$$i = \mathbf{17.471}\%$$

5. Find the present value of a perpetuity-immediate that makes the following payments:

Time	0	1	2	3	4	5	6	7	8	⋯
Payment		200	180	160	140	120	100	80	80	⋯

   After time 7, this perpetuity continues to pay 80 forever. The effective interest rate is $i = 6\%$.

Time	0	1	2	3	4	5	6	7	8	⋯
Level Perpetuity		80	80	80	80	80	80	80	80	⋯
Decreasing Annuity		120	100	80	60	40	20			

$$80a_{\overline{\infty}|} + 20(Da)_{\overline{6}|}$$
$$= \frac{80}{0.06} + 20\frac{6 - a_{\overline{6}|}}{0.06} = \mathbf{1694.23}$$

6. A 10-year decreasing annuity-due makes a payment of 100 at the beginning of the first year, and each subsequent payment is 5 less then the previous. What is the accumulated value of this annuity at time 10 (one year after the final payment)? The effective annual rate of interest is 5%. Note, the final payment is 55, not 5.

These are two of many possible solutions.
Solution 1:

Time	0	1	2	3	4	5	6	7	8	9	10
Level Annuity	50	50	50	50	50	50	50	50	50	50	
Decreasing Annuity	50	45	40	35	30	25	20	15	10	5	

The accumulated value of this annuity at time 9 is

$$50s_{\overline{10}|} + 5(Ds)_{\overline{10}|}$$

Thus the accumulated value of this annuity at time 10 is

$$\left(50s_{\overline{10}|} + 5(Ds)_{\overline{10}|}\right)(1.05)$$

$$= \left[50s_{\overline{10}|} + 5\left(\frac{10(1.05)^{10} - s_{\overline{10}|}}{.05}\right)\right](1.05) = \mathbf{1050}$$

Solution 2:

Time	0	1	2	3	4	5	6	7	8	9	10
Level Annuity	100	100	100	100	100	100	100	100	100	100	
Increasing Annuity		-5	-10	-15	-20	-25	-30	-35	-40	-45	

The accumulated value of this annuity at time 9 is

$$100s_{\overline{10}|} - 5(Is)_{\overline{9}|}$$

Thus the accumulated value of this annuity at time 10 is

$$\left(100s_{\overline{10}|} - 5(Is)_{\overline{9}|}\right)(1.05)$$

$$= \left[100s_{\overline{10}|} - 5\left(\frac{s_{\overline{10}|} - 10}{.05}\right)\right](1.05) = \mathbf{1050}$$

7. A 5-year annuity makes payments continuously at a rate $h(t) = e^{\left(\frac{t^2}{24}+t\right)}$. The force of interest varies continuously at a rate of $\delta_t = \frac{t}{12}$. Find the present value of this annuity.

$$\int_0^5 e^{\left(\frac{t^2}{24}+t\right)} e^{-\int_0^t \frac{s}{12}\,ds}\,dt$$

$$= \int_0^5 e^{\left(\frac{t^2}{24}+t\right)} e^{\left(-\frac{t^2}{24}\right)}\ dt$$
$$= \int_0^5 e^t\ dt$$
$$= e^5 - 1$$
$$= \mathbf{147.41}$$

For problems 8, 9, and 10, Sam borrows \$5000 from the bank at an effective annual interest rate $i = 10\%$. He will repay the loan with payments at the end of each year for 4 years. Create an amortization schedule for the loan similar to Tables 3.3 and 3.5 in Broverman. In each problem, indicate the total amount of all payments made, the total amount of all interest paid, and the total amount of all principal repaid.

8. Create an amortization schedule where Sam repays the loan with 4 level payments.

$$k\, a_{\overline{4}|} = 5000$$

$$k = \frac{5000}{a_{\overline{4}|}} = \frac{5000}{3.1699} = 1577.35$$

t	*Payment*	*Interest Due*	*Principal Repaid*	*Outstanding Balance*
0				5000.00
1	1577.35	500.00	1077.35	3922.65
2	1577.35	392.26	1185.09	2737.56
3	1577.35	273.76	1303.60	1433.96
4	1577.35	143.40	1433.96	0
Total	6309.42	1309.42	5000.00	

9. Create an amortization scedule where the first payment is k, and each subsequent payment is 5% larger then the preceding payment.

$$k\left(\frac{1-\left(\frac{1.05}{1.10}\right)^4}{.10-.05}\right) = 5000$$

$$3.39586k = 5000$$

$$k = 1472.38$$

t	*Payment*	*Interest Due*	*Principal Repaid*	*Outstanding Balance*
0				5000.00
1	1472.38	500.00	972.38	4027.62
2	1546.00	402.76	1143.24	2884.38
3	1623.30	288.44	1334.86	1549.51
4	1704.47	154.95	1549.51	0
Total	6346.15	1346.15	5000.00	

10. Create an amortization schedule where the first payment is k, and each subsequent payment is \$200 larger then the preceding.

$$k\, a_{\overline{4}|} + 200\frac{a_{\overline{4}|} - 4v^4}{i} = 5000$$

$$k \cdot 3.1699 + 875.62 = 5000$$

$$k = 1301.12$$

t	*Payment*	*Interest Due*	*Principal Repaid*	*Outstanding Balance*
0				5000.00
1	1301.12	500.00	801.12	4198.88
2	1501.12	419.89	1081.23	3117.65
3	1701.12	311.76	1389.36	1728.29
4	1901.12	172.83	1728.29	0
Total	6404.48	1404.48	5000.00	