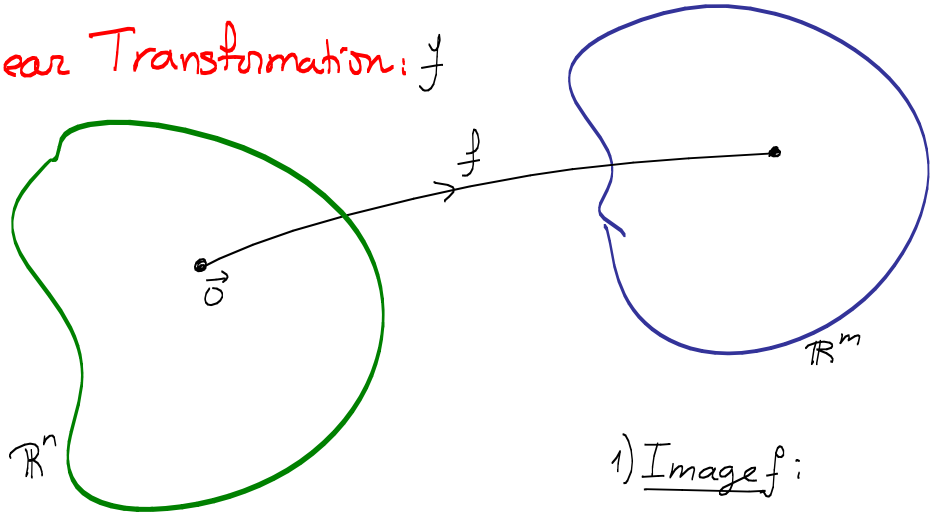
Subspaces associated with a Linear transformation

Linear Transformation: f

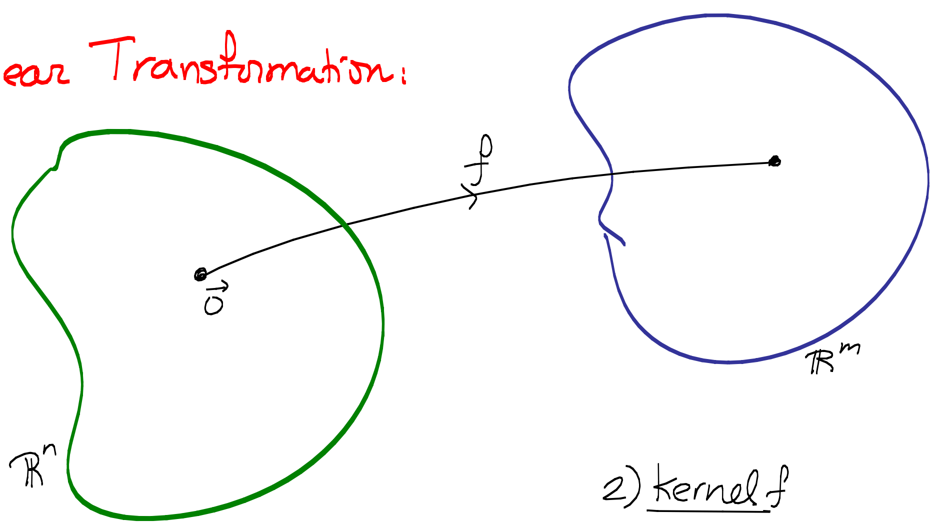


1) <u>Image</u> f:

$\vec{x} \mapsto A\vec{x}$

Subspaces associated with a Linear transformation

Linear Transformation:



2) <u>kernel</u> f

$\vec{x} \mapsto A\vec{x}$

For a function $\mathbb{R} \to \mathbb{R}$

$f(x) = \cos x$, Image f =

$f(x) = x^2$: Image f =

For a function $f: \mathbb{R} \to \mathbb{R}^2$

$f(x) = \begin{pmatrix} x^2 \\ x \end{pmatrix}$ image f =

For a linear map,

$f(\vec{x}) = A\vec{x}$

<u>Defn</u>: $\operatorname{Im} f = \{ \vec{b} : A\vec{x} = \vec{b}$ is <u>consistent</u> $\}$.

<u>Example</u>: $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Image = ?

Example: $T: \mathbb{R}^2 \to \mathbb{R}^3$

WS #1

$$\vec{x} \mapsto \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}\vec{x}$$

Example: $T: \mathbb{R}^2 \to \mathbb{R}^3$

$$\vec{x} \mapsto \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix} \vec{x}$$

WS#2

For a matrix $A = \begin{pmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_r \\ | & | & & | \end{pmatrix}$

Column space A = span$\{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_r\}$

Example: $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

Defn Span$\{\vec{a}_1, \ldots, \vec{a}_r\}$ =

Theorem The image of $T(\vec{x}) = A\vec{x}$ is the column space of A.

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_r\vec{a}_r$$

How to find the image?

$$A = \begin{pmatrix} | & | & & | \\ \vec{a}_1, & \vec{a}_2, & \cdots & \vec{a}_n \\ | & | & & | \end{pmatrix} \underset{\text{row reduction}}{\sim} \begin{pmatrix} 1 & \text{---} & & & \\ 0 & 1 & \text{---} & & \\ 0 & 0 & 0 & 1 & \text{---} \end{pmatrix}$$

Example

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 3 \end{pmatrix}$$

$\operatorname{Im} A =$

Kernel: (No analogue in calculus)

$T : \mathbb{R}^n \to \mathbb{R}^m$

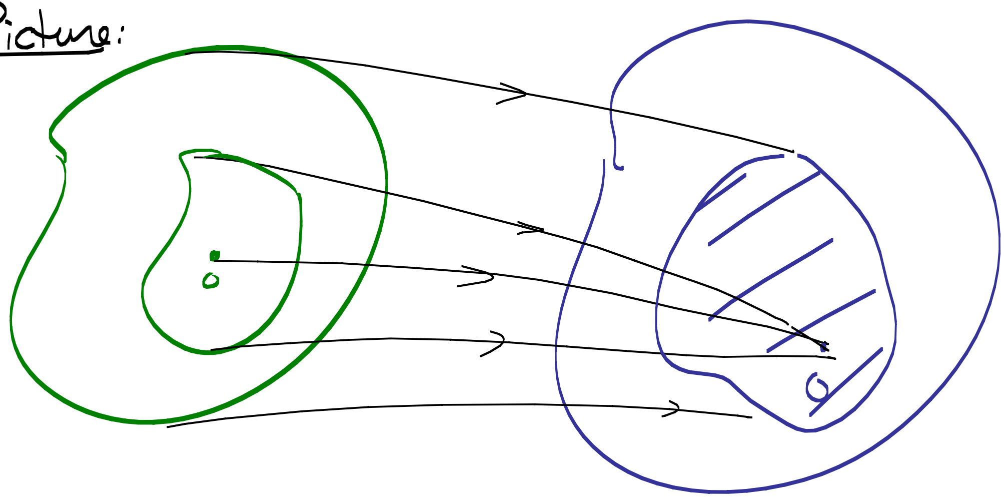
$T(x) = Ax$

Example: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ projection onto x-axis

Example: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Defn The kernel of a linear map $\mathbb{R}^n \to \mathbb{R}^m$

Picture:



How to find the kernel

Solve: $A\vec{x} = \vec{0}$

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{pmatrix} \sim$$

Solutions:

WS#3.

Properties of image and kernel:

1) $\vec{0}$ is in both image and kernel

2) if $\vec{x} \in \begin{cases} \ker T \\ \operatorname{Im} T \end{cases}$ then $a\vec{x} \in \begin{cases} \ker T \\ \operatorname{im} T \end{cases}$

3) If $\vec{x}, \vec{y} \in \begin{cases} \ker T \\ \operatorname{Im} T \end{cases}$ then $\vec{x} + \vec{y} \in \begin{cases} \ker T \\ \operatorname{Im} T \end{cases}$

Q1: When is the image of A all of $\mathbb{R}^m$?

Q2: When is the kernel of A only $\{\vec{0}\}$?