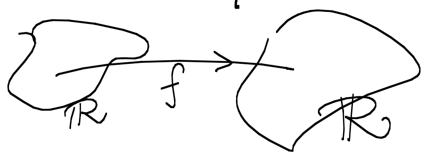
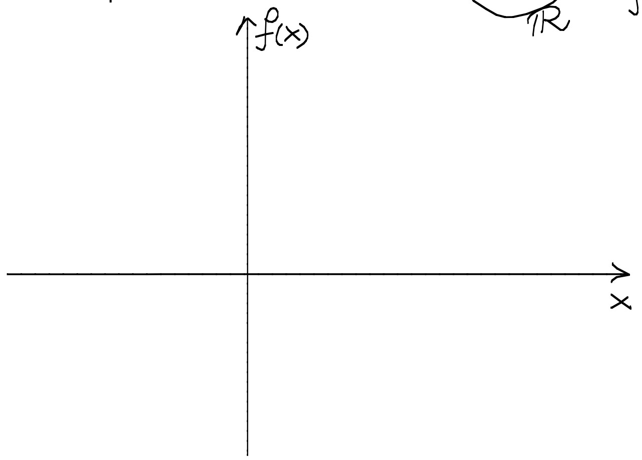
Math 415 2/2/12 Invertibility

Maps $\mathbb{R} \to \mathbb{R}$

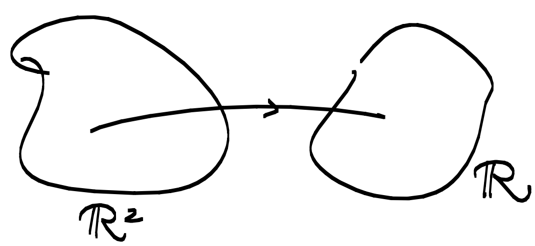




- $f(x) = x^2, \quad x \in [-1,1]$
- $f(x) = \cos x, \quad x \in [0, \frac{\pi}{2}]$
- $f(x) = \cos x, \quad x \in [0, \pi]$
- $f(x) = 2x+1, \quad x \in [0,1]$
- $f(x) = 0, \quad x \in [-1,1]$

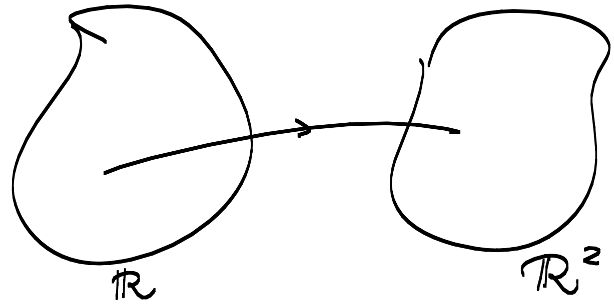
Invertible?

Maps $\mathbb{R}^2 \to \mathbb{R}$



- $f\begin{pmatrix} x \\ y \end{pmatrix} = x + y$
- $f\begin{pmatrix} x \\ y \end{pmatrix} = 2x$
- $f\begin{pmatrix} x \\ y \end{pmatrix} = xy$
- $f(x,y) = 0$

Maps $\mathbb{R} \to \mathbb{R}^2$



- $f(x) = \begin{pmatrix} x \\ x \end{pmatrix}$
- $f(x) = \begin{pmatrix} 2x \\ x \end{pmatrix}$
- $f(x) = \begin{pmatrix} 0 \\ x \end{pmatrix}$
- $f(x) = \begin{pmatrix} x+1 \\ x \end{pmatrix}$
- $f(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Maps $\mathbb{R}^2 \to \mathbb{R}^2$

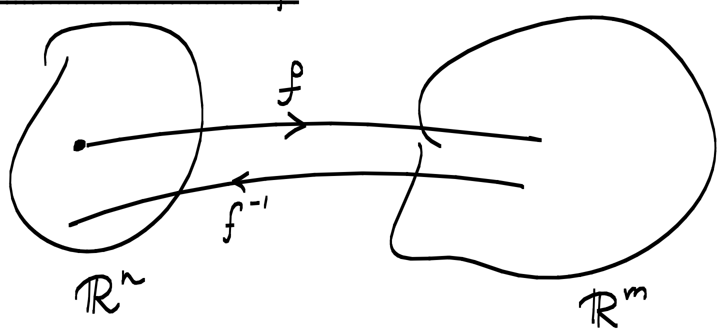
- $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 3y \end{pmatrix}$
- $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$
- $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+3y \\ x-2y \end{pmatrix}$
- $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ 4x+2y \end{pmatrix}$
- $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ x+y \end{pmatrix}$
- $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Example: Rotations in $\mathbb{R}^2$?

Example: reflections in $\mathbb{R}^2$?

Example: Projections in $\mathbb{R}^2$?

Inverse of a map:



Example: 1) $f(x) = 2x \Rightarrow f^{-1}(y) =$

2) $f(x) = 0 \Rightarrow f^{-1}(y) =$

3) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} \Rightarrow f^{-1}\begin{pmatrix} 1 \\ 3 \end{pmatrix} =$

4) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix} \Rightarrow f^{-1}\begin{pmatrix} 2 \\ 0 \end{pmatrix} =$

Defn: the map $f: D \to R$ has a left inverse $g: R \to D$
if $g(f(x)) = x$ for each x in D

the map $f: D \to R$ has a right inverse $h: R \to D$
if $f(g(y)) = y$ for each y in R

Example: $f(x) = \begin{pmatrix} 2x \\ x \end{pmatrix}$ $(\mathbb{R} \to \mathbb{R}^2)$

$g\begin{pmatrix} x \\ y \end{pmatrix} = y$ is a left inverse of f

$$x \xmapsto{f} \begin{pmatrix} 2x \\ x \end{pmatrix} \xmapsto{g} x$$

But

$$\begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{g} y \longmapsto \begin{pmatrix} 2y \\ y \end{pmatrix} \neq \begin{pmatrix} x \\ y \end{pmatrix}$$

<u>Defn</u>: a map $f: D \to R$ has an inverse $g: R \to D$ if g is a left and right inverse of f.

<u>$f(x): \mathbb{R}^3 \to \mathbb{R}^2$</u> $\qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = Ax$

1) Find a 3×2 matrix B : $AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so that

$$A\left(B\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

e.g. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$

? 2) Find a 3×2 matrix C : $CA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow C\left(A\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

A linear map $f: \mathbb{R}^n \to \mathbb{R}^m$ is invertible $\Rightarrow n=m$

Not all linear maps $\mathbb{R}^n \to \mathbb{R}^n$ are invertible!

Example: $\vec{x} \mapsto A\vec{x}$, $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

find B: $\quad BA\vec{x} = \vec{x}$

$$B \begin{pmatrix} x_1 + x_2 \\ 2(x_1+x_2) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

1) A matrix A is invertible $\Leftrightarrow$

$A \sim \mathbb{1}$ in reduced row echelon form.

2) In that case, the inverse is found by

$$(A \mid \mathbb{1}) \sim (\mathbb{1} \mid B) \Rightarrow B = A^{-1}.$$

Example 1: $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Example 2: $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

Example 3:

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

Example 4:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Example 5: $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$

Example 6: $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$

Example 7: A, B invertible: $(AB)^{-1} = ?$

Example 8: If $A^2 = \mathbb{1}$, $A^{-1} = ?$

Example 9: If $A^m = \mathbb{1}$, $A^{-1} = ?$

Example 10 If $A^2 = A$, $A^{-1} = ?$

If A has an inverse

The Linear system

$$A\vec{x} = \vec{b}$$

Has a solution

Has a unique solution.