Selected problem solutions 2.3, 2.5

2.3

(17) Find all matrices (2×2) which commute with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Sol: The matrix A rescales the second row of B in the product $AB = \begin{pmatrix} b_{11} & b_{12} \\ 2b_{21} & 2b_{22} \end{pmatrix}$

and it rescales the second column in the product

$$BA = \begin{pmatrix} b_{11} & 2b_{12} \\ b_{21} & 2b_{22} \end{pmatrix}$$

Therefore we must have

$$2b_{21} = b_{21} \Rightarrow b_{21} = 0$$

and $\quad 2b_{12} = b_{12} \Rightarrow b_{12} = 0.$

Then $\left\{ B = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \alpha, \beta \in \mathbb{R} \right\}$ commute with A, that is, the set of all diagonal 2×2 matrices.

29) $D_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ corresponds to a counterclockwise rotation in $\mathbb{R}^2$.

a) The product of two rotations, $D_\alpha D_\beta$, commute in $\mathbb{R}^2$: It doesn't matter in which order we rotate. Therefore, $D_\alpha D_\beta = D_\beta D_\alpha = D_{\alpha+\beta}$.

b) To prove this algebraically compute

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \sin\beta\cos\alpha & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} \quad \text{(use trig identities)}$$

$$= \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

(since the product matrix is symmetric under the exchange $\alpha \leftrightarrow \beta$) $\square$

32) Find all 2×2 matrices which commute with every 2×2 matrix.

Solution: This is the set of all matrices $\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \alpha \in \mathbb{R} \right\}$

Proof: It is clear that $(\alpha \mathbb{1})A = \alpha(\mathbb{1}A) = \alpha(A\mathbb{1}) = \alpha A$
$= A(\alpha\mathbb{1})$

So $\alpha\mathbb{1} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ commutes with every matrix.

To show that there are no other such matrices consider

$\beta c = \gamma b \quad \forall b,c \Rightarrow \beta = \gamma = 0$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a + \beta c & \alpha b \\ & \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha a + \gamma b & \delta b \\ * & * \end{pmatrix}$$

$\alpha = \delta$.

D.

36) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

Proof: by induction. True for $n=1$. Suppose true for $n-1$. Then

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{n-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \underset{\uparrow}{=} \begin{pmatrix} 1 & n-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

by induction hypothesis

$$= \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

(Represents a "shear" matrix).

43) Find a matrix $A \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ s.t. $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

<u>Example</u>: 1) $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

2) $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

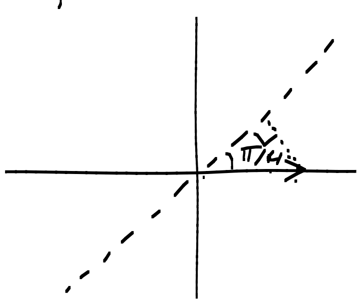
3) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ etc.

44) Find a matrix A whose square is not the identity matrix but whose 4^{th} power is.

<u>Solution</u>: Think of a rotation by $\frac{\pi}{2}$. Then $A^4 = \mathbb{1}$.
The matrix representing this rotation is

$$A = \begin{pmatrix} \cos\pi/2 & -\sin\pi/2 \\ \sin\pi/2 & \cos\pi/2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

46) Find a matrix A: $A^2 = A$ and all $a_{ij} \neq 0$.
This is a projection matrix. For example, projection onto the $x=y$ line



$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A^2 = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = A$$

47) Find a 2×2 matrix A s.t. $A^3 = A$ and all the entries of A are non-zero.

Solution: Any projection matrix has $A^2 = A$ hence $A^3 = A(A^2) = A \cdot A = A^2 = A$.

Therefore the solution to #46 works:

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

48) Find a matrix such that $A^{10} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Solution: The proof that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ can be adapted to show that $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n\alpha \\ 0 & 1 \end{pmatrix}$.

In fact,
$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha+\beta \\ 0 & 1 \end{pmatrix}$$

So $A = \begin{pmatrix} 1 & 0.1 \\ 0 & 1 \end{pmatrix}$ works: $A^{10} = \begin{pmatrix} 1 & 10(0.1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Section 2.4

5) Find the inverse (if it exists) of the matrix $\begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

Solution: Write

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right)$$

Since there is no pivot in the third row, the matrix is not invertible

6) The matrix $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ is invertible:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$\Rightarrow$ the inverse is $\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$. Check:

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

8) The inverse of the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is itself: It corresponds to a permutation $1 \leftrightarrow 3$. Check:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

10) The matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$ is invertible, with inverse $\begin{pmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix}$

Compute:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 6. & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

Check: $\begin{pmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

29) For which values of k is the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{pmatrix}$ invertible? Check that the reduced form has a pivot in all rows:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & k-1 \\ 0 & 3 & k^2-1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & k-1 \\ 0 & 0 & k^2-1-3(k-1) \end{pmatrix}$$

If k is not a root of $\underbrace{k^2-3k+2}_{k\neq -1,-2}$, matrix invertible.

34) The matrix $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ is invertible if and only if the product $abc \neq 0$. That is, each of a, b, c is nonzero. The proof is by row-reduction: If any of them is zero, then there is no pivot in that row.

The inverse is $\begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & c^{-1} \end{pmatrix}$.

35) The triangular matrix $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$ is again invertible iff $adf \neq 0$. In that case there is a pivot in each row. The inverse is computable:

$$\left(\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ 0 & d & e & 0 & 1 & 0 \\ 0 & 0 & f & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & \frac{b}{a} & \frac{c}{a} & a^{-1} & 0 & 0 \\ 0 & 1 & \frac{e}{d} & 0 & d^{-1} & 0 \\ 0 & 0 & 1 & 0 & 0 & f^{-1} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & b/a & 0 & a^{-1} & 0 & -\frac{c}{af} \\ & 1 & 0 & 0 & d^{-1} & -\frac{e}{df} \\ & & 1 & & & f^{-1} \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & a^{-1} & \left(-\frac{b}{ad}\right) & \left(\frac{-c}{af} - \frac{be}{adf}\right) \\ 0 & 1 & 0 & 0 & d^{-1} & \left(-e/df\right) \\ 0 & 0 & 1 & 0 & 0 & f^{-1} \end{array}\right).$$

or it can be computed by multiplication.

38) The inverse of $A = \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix}$ is $A^{-1} = \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix} = A$

pf: $\begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

41) Which of the following $T: \mathbb{R}^3 \to \mathbb{R}^3$ are invertible:

a) reflection about a plane: invertible. The inverse is the same reflection.

b) Orthogonal projection onto a plane: not invertible. Example: $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. the information about the z-coordinate is lost.

c) Scaling by a factor of 5 is invertible: The inverse is scaling by a factor $\frac{1}{5}$.

d) Rotation about an axis is invertible: a rotation in the opposite direction is the inverse.

42) Permutations are always invertible: the inverse is another permutation.

43) The inverse of the transformation $T(x) = AB\vec{x}$ is the transformation $B^{-1}A^{-1}\vec{x}$ if A, B are invertible.

Proof: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(\mathbb{1})A^{-1}$

$= AA^{-1} = \mathbb{1}$.