

# The Hilbert Matrix and its Determinant

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## Abstract

The Hilbert matrix is  $A_n \doteq (\frac{1}{i+j+1})$  for  $0 \leq i, j \leq n-1$ . We show how to evaluate  $\det(A_n)$ .

The Hilbert Matrix defined above is the matrix  $\langle x^i, x^j \rangle$  for  $0 \leq i, j \leq n-1$  where the inner product is given by

$$\langle f, g \rangle \doteq \int_0^1 f(t)g(t)dt.$$

Some notation. Of course  $n! = \prod_i^n i$ . Define  $n! = \prod_1^n i! = n(n-1)^2(n-2)^3 \dots 2^{n-1}1^n$ .

**Theorem 1** *Suppose*

$$A_n = (\frac{1}{i+j+1})_{0 \leq i, j \leq n-1}.$$

*Then*

$$\det(A) = \frac{((n-1)!)^4}{(2n-1)!}.$$

**Proof:** The proof is by induction on  $n$ . When  $n = 1$  we have

$$A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and then  $\det(A) = \frac{1}{1}$  which is the desired form for  $n = 1$ .

Suppose the result is true for  $n-1$  and consider  $\det(A_n)$ .

$$\mathbf{A}_n = \begin{pmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n-1} & \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-2} \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{pmatrix} \tag{1}$$

Subtract the last ( $n^{\text{th}}$ ) row from each row above it. The entry in the  $j^{\text{th}}$  column of the  $i^{\text{th}}$  row is

$$\frac{1}{i+j-1} - \frac{1}{n+j-1} = \frac{n-i}{(i+j-1)(n+j-1)}.$$

The new matrix is then

$$B = \begin{pmatrix} \frac{n-1}{(1)(n)} & \frac{n-1}{(2)(n+1)} & \frac{n-1}{(3)(n+2)} & \cdots & \frac{n-1}{(n)(2n-1)} \\ \frac{n-2}{(2)(n)} & \frac{n-2}{(3)(n+1)} & \frac{n-2}{(4)(n+2)} & \cdots & \frac{n-2}{(n+1)(2n-1)} \\ \frac{n-3}{(3)(n)} & \frac{n-3}{(4)(n+1)} & \frac{n-3}{(5)(n+2)} & \cdots & \frac{n-3}{(n+3)(2n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{n-(n-1)}{(n-1)(n)} & \frac{n-(n-1)}{(n)(n+1)} & \frac{n-(n-1)}{(n+1)(n+2)} & \cdots & \frac{n-(n-1)}{(2n-2)(2n-1)} \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{pmatrix}$$

We know that  $\det(A_n) = \det(B)$ . To compute  $\det(B)$  we see that we can factor the term  $n-i$  from the  $i^{\text{th}}$  row for  $1 \leq i < n$  and  $\frac{1}{n+j-1}$  from the  $j^{\text{th}}$  column for  $1 \leq j \leq n$ . Hence we get that

$$\det(A_n) = \frac{(n-1)!^2}{(2n-1)!} \det \begin{pmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n-1} & \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-2} \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Now subtract the last column from the preceding columns. The resulting matrix is

$$B = \begin{pmatrix} \frac{n-1}{(1)(n)} & \frac{n-2}{(2)(n)} & \frac{n-3}{(3)(n)} & \cdots & \frac{1}{n} \\ \frac{n-1}{(2)(n+1)} & \frac{n-2}{(3)(n+1)} & \frac{n-3}{(4)(n+1)} & \cdots & \frac{1}{n+1} \\ \frac{n-1}{(3)(n+2)} & \frac{n-2}{(4)(n+2)} & \frac{n-3}{(5)(n+2)} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{n-1}{(n-1)(2n-2)} & \frac{n-2}{(n)(2n-2)} & \frac{n-3}{(n+1)(2n-2)} & \cdots & \frac{1}{2n-2} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

As before we can now factor  $n-j$  from the  $j^{\text{th}}$  column and  $\frac{1}{n+i+1}$  from the  $i^{\text{th}}$  row for  $1 \leq i, j \leq n-1$  getting

$$\det(A_n) = \frac{(n-1)!^4}{(2n-1)!(2n-2)!} \det \begin{pmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n-1} & 1 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n} & 1 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+1} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-3} & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Hence

$$\det(A_n) = \frac{(n-1)!^4}{(2n-1)!(2n-2)!} \det(A_{n-1}).$$

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The formula in the statement follows.

So, for example

$$\begin{aligned}\det A_1 &= \frac{1}{1} \\ \det A_2 &= \frac{1}{12} \\ \det A_3 &= \frac{1}{2160} \\ \det A_4 &= \frac{1}{6048000} \\ \det A_5 &= \frac{1}{266716800000} \\ \det A_6 &= \frac{1}{186313420339200000} \\ \det A_7 &= \frac{1}{2067909047925770649600000} \\ \det A_8 &= \frac{1}{365356847125734485878112256000000} \\ \det A_9 &= \frac{1}{1028781784378569697887052962909388800000000} \\ \det A_{10} &= \frac{1}{46206893947914691316295628839036278726983680000000000} \\ \det A_{11} &= \frac{1}{33122504897063413755362143627040727106080127672469422080000000000}\end{aligned}$$

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