

Practice Hour Examination

Answers

6 March 2008

Problem 1 *The orthonormal unit base vectors in spherical coordinates are*

$$\mathbf{e}_\rho = \begin{pmatrix} \sin \phi \cos \theta \\ \sin \phi \sin \theta \\ \cos \phi \end{pmatrix}, \mathbf{e}_\phi = \begin{pmatrix} \cos \phi \cos \theta \\ \cos \phi \sin \theta \\ -\sin \phi \end{pmatrix}, \mathbf{e}_\theta = \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix}$$

in terms of the cartesian coordinates. Write the cartesian coordinate orthonormal base vectors as linear combinations of these spherical coordinate base vectors.

Proof: The matrix that gives the one basis in terms of the other is orthogonal and hence its inverse is its transpose. Since

$$(\mathbf{e}_\rho \quad \mathbf{e}_\phi \quad \mathbf{e}_\theta) = (\mathbf{e}_1 \quad \mathbf{e}_1 \quad \mathbf{e}_1) \begin{pmatrix} \sin \phi \cos \theta & \cos \phi \cos \theta & -\sin \theta \\ \sin \phi \sin \theta & \cos \phi \sin \theta & \cos \theta \\ \cos \phi & -\sin \phi & 0 \end{pmatrix} \quad (1)$$

Then

$$(\mathbf{e}_1 \quad \mathbf{e}_1 \quad \mathbf{e}_1) = (\mathbf{e}_\rho \quad \mathbf{e}_\phi \quad \mathbf{e}_\theta) = \begin{pmatrix} \sin \phi \cos \theta & \sin \phi \sin \theta & \cos \phi \\ \cos \phi \cos \theta & \cos \phi \sin \theta & -\sin \phi \\ -\sin \theta & \cos \theta & 0 \end{pmatrix} \quad (2)$$

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Problem 2 *Find the eigenvalues and normal eigenvectors for the matrix*

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 4 & 3 \\ -1 & 3 & 4 \end{pmatrix}$$

Verify that the eigenvectors are orthogonal to each other.

Proof: Since

$$\det(A) = 0 \quad (3)$$

it follows that $\lambda_3 = 0$ is an eigenvalue and that the characteristic polynomial

$$\det(XI - A) = X(X^2 - 10X + E) \quad (4)$$

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where E is an integer. In calculating the determinant, it is seen that $E = 21$ and hence the other two eigenvalues are $\lambda_1 = 7, \lambda_2 = 3$. To find the eigenvectors we want to solve the three equations

$$(A - \lambda_i I) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (5)$$

It is easy to see that the three solutions are

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (6)$$

for the eigenvalues 7, 3, 0 respectively.

These are mutually orthogonal by inspection. To get normal eigenvectors, just divide each by its norm, namely $\sqrt{2}, \sqrt{6}, \sqrt{3}$ respectively. ■

Problem 3 Suppose that \mathbf{u}, \mathbf{v} are orthonormal vectors in \mathbb{R}^3 . If

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

demonstrate that

$$\mathbf{w} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ -(u_1 v_3 - u_3 v_1) \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

is a normal vector.

Proof: Use the fact that

$$u_1^2 + u_2^2 + u_3^2 = 1 \quad (7)$$

$$v_1^2 + v_2^2 + v_3^2 = 1 \quad (8)$$

$$u_1 v_1 + u_2 v_2 + u_3 v_3 = 0 \quad (9)$$

to show that

$$(u_2 v_3 - u_3 v_2)^2 + (u_1 v_3 - u_3 v_1)^2 + (u_1 v_2 - u_2 v_1)^2 = 1 \quad (10)$$

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Problem 4 Find necessary and sufficient conditions in order that the matrix V in $\mathbb{R}^{3 \times 3}$ have an inverse. When it has an inverse, find it.

$$V = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} \quad (11)$$

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Proof: Necessary and sufficient conditions are that $\det(V) \neq 0$. Since $\det(V) = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$ this is the same as to say that the λ_i are distinct. When this is the case, then the inverse is given by the adjoint divided by the determinant. That is

$$V^{-1} = \frac{1}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \begin{pmatrix} \lambda_3 - \lambda_2 & 0 & 0 \\ 0 & \lambda_3 - \lambda_1 & 0 \\ 0 & 0 & \lambda_2 - \lambda_1 \end{pmatrix} \begin{pmatrix} \lambda_2 \lambda_3 & -(\lambda_2 + \lambda_3) & 1 \\ -\lambda_1 \lambda_3 & \lambda_1 + \lambda_3 & -1 \\ \lambda_1 \lambda_2 & -(\lambda_1 + \lambda_2) & 1 \end{pmatrix} \quad (12)$$

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