MATH 487 MM : Mathematical Methods in ENGINEERING

Argument of the Course

Problem: To be successful in upper-division engineering courses, it is necessary to know how to use linear algebra and calculus. Therefore, all engineering students take calculus. Much to our dismay, it does not follow that all engineering students are prepared to *use* calculus as a comprehensive tool in applications.

All the calculus, linear algebra, and differential equations swimming in your head will be revived, extended, and integrated into a machine for attacking engineering problems.

Method: There are three foci about which this course will revolve: (1) linear algebra; (2) differential equations; (3) complex variables. Throughout the course, we will build on prior results and teach methods in the context of applications you are likely to encounter in later ECE coursework.

Outline of mathematics topics: (subject to modification)

Emphasis I: linear algebra

1. Euclidean geometry: vectors, dot products, R^n
2. Matrix algebra: matrices, solving equations, eigendecompositions
3. Calculus: derivatives, change of coordinates

Emphasis II: differential equations

1. Linear systems: matrix exponentials
2. Nonlinear equations: fixed points and linearization
3. PDEs: separation of variables

Emphasis III: complex variables

1. Fourier theory: transforms, applications to ODEs, PDEs
2. Contour intergration: Cauchy-Riemann, poles
3. More general integration: Differential operators, Green/Gauss/Stokes

Lecture 1: Euclidean geometry

1. definition of R^n
2. why you should care about working in R^n for $n > 3$
3. the difference between points and vectors
4. definition of *span*
5. the *standard basis* of R^n and uniqueness of representation

Lecture 2: Operations on vectors

1. definition of *dot products*
2. definition of the *norm* of a vector in terms of dot products
3. recall: notion of *span*; examples
4. observation: a span is a *subspace* of R^n
5. definition of *linear independence*
6. definition of *basis*
7. examples

Lecture 3: Coordinates

1. definition of *coordinates* in a given basis
2. lemma: coordinates are unique
3. problem 1: given a basis and a vector in standard coordinates, what are its coordinates in the basis?
4. observation: this can be written as a matrix-vector problem
5. problem 2: given two bases, convert coordinates of a vector from basis to the other
6. observation: this can be written as a matrix problem
7. definitions: *orthogonal* and *orthonormal* bases
8. derivation of dot-product formula for ON basis coordinates
9. definition: *orthogonal matrix*
10. observation: it's easy to compute the inverse of an orthogonal matrix

Lecture 4: Matrix algebra

1. ways to do matrix-vector and matrix-matrix multiplication
2. the identity matrix
3. powers of matrices
4. caveats; noncommutativity, noncancellation, etc.
5. special matrices; diagonal, triangular, symmetric
6. transpose
7. definition: *row operations*
8. determinants via row operations
9. facts: determinants and transposes, products, and triangular matrices
10. simple formula for a 2-by-2 determinant

Lecture 5: Determinants and inverses

1. geometric interpretation of determinants in terms of volumes
2. examples of computing determinants via row reduction
3. definition of the inverse of a matrix

4. determinants of inverses
5. application of inverses to change of basis formula
6. theorem: invertible iff determinant is nonzero

Lecture 6: Inverses

1. problem: how to compute a matrix inverse
2. procedure: row reduction of augmented matrix
3. examples
4. definition: *rank*
5. corollary: invertibility in terms of rank

Lecture 7: Derivatives

1. intuition: the derivative of $f : R \to R$ as a rate of change
2. intuition: the derivative of $f : R \to R^n$ should measure n rates of change (geometric intuition — curve)
3. intuition: the derivative of $f : R^m \to R$ should measure a rate of change with respect to m inputs (geometric intuition — level sets)
4. intuition: whatever the derivative of $f : R^m \to R^n$ is, is should measure *rates of change* and be *linear*
5. fact: $[Df(a)]$ is a matrix for each a in domain
6. picture: $[Df]$ as a linear transformation on tangent vectors
7. example: electric circuit with lots of resistors/currents
8. *chain rule* in terms of matrix multiplication
9. examples of 'classical' chain rule statements

Lecture 8: Coordinates and derivatives

1. more examples of the chain rule
2. the derivative of a power of $f : R^n \to R^n$
3. polar coordinates in the plane: $P : (r, \theta) \mapsto (x, y)$
4. how does P change the basis vectors $\hat{x}, \hat{y}$?
5. applications of derivatives to integrals: $\det[DP] = r$
6. general change of coordinates formula for integrals
7. why this is the same thing as 'u-substitution'

Lecture 9: More coordinates

1. polar, cylindrical, spherical coordinates
2. computation of determinants of derivatives
3. application: change of basis formulae
4. application: integration
5. other coordinate systems and integration
6. example: computing the area of an ellipse via rescaled coordinates
7. example: computing the area of a Carnot cycle

Lecture 10: Eigendecompositions

1. definition: eigenvalues, eigenvectors
2. many examples
3. application: anisotropy

Lecture 11: Eigendecompositions

1. the case of all real, distinct eigenvalues
2. application: diagonalizing matrices
3. application: powers of matrices
4. examples: repeated and complex eigenvalues
5. intro to generalized eigenvectors

Lecture 12: Linear ODE systems

1. recollection of second-order ODEs
2. conversion to first-order systems
3. matrix form of first-order systems
4. general solution in case of real, distinct roots
5. example: general 2-by-2 system
6. example: initial value problems
7. observation: this seems related to the problem of diagonalizing a matrix

Lecture 13: Matrix exponentials

1. definition: e^A, e^{At}
2. theorem: $e^{At}x_0$ solves $x' = Ax$ with $x(0) = x_0$
3. computing e^{At} via diagonalization
4. examples

Lecture 14: More matrix exponentials

1. lemma: if $AB = BA$ then $e^{A+B} = e^A e^B$
2. computation: e^{At} where $A = [\lambda, 1; 0, \lambda]$
3. general case of repeated real eigenvalues
4. simple examples with complex eigenvalues
5. (Sylvester's formulae if there's time...)
6. (general mumbling about Jordan forms...)

Lecture 15: Examples and extensions

1. example 1: an RLC circuit
2. example 2: a 3-by-3 system with complex eigenvalues
3. inhomogeneous systems: general (matrix) solution
4. example, with the moral that this equation is not pleasant

Lecture 16: Nonlinear systems

1. examples and importance of nonlinear systems of ODEs
2. definition: fixed (equilibrium) points
3. definition: linearized system at a fixed point
4. Hartman-Grobman Theorem
5. example: $x' = \sin x$ on R; compare analytic versus qualitative solutions
6. example: Lotka-Volterra model

Lecture 17: Phase plane analysis

1. derivation of trace-determinant diagram for 2-by-2 linear system
2. application to nonlinear 2-d systems
3. determination of stability via eigenvalues
4. examples of neutrally stable fixed points
5. (mumbling about higher dimensional phase space and chaos)

Lecture 18: Intro to PDEs

1. the heat (diffusion) equation
2. lemma: this is a linear equation
3. examples of solutions
4. form of examples motivates the separation of variables ansatz
5. separation of variables; fixed boundary conditions
6. (mumble about the initial condition and Fourier...)

Lecture 19: Intro to PDEs II

1. example: heat equation with insulating/mixed boundaries
2. example: wave equation with fixed boundaries
3. example: Laplace equation
4. (skip initial conditions and mumble about Fourier...)

Lecture 20: Fourier expansions

1. recollection of Taylor expansions
2. use initial value problem for heat/wave to motivate Fourier expansions
3. examples with odd and even functions
4. lemma: odd (even) functions have pure sin (cos) expansions
5. formulae for sin and cos expansions
6. applications to initial value problems in PDEs
7. Euler formula for cos/sin integrals
8. (mumble about orthogonal basis functions and inner products of functions)

Lecture 21: Fourier transform

1. discuss unitary transforms in R^n as complex generalization of orthogonal matrices (not in Greenberg)
2. example: DFT, show explicitly that $U^*U = UU^* = I$
3. wave hands about transition to infinite dimensional spaces
4. show formal analogy between integrals and matrix mult.
5. introduce FT as unitary transform.
6. change variables to put in Greenberg form (not symmetric normalization)

Lecture 22: More FT

1. scaling, shift, linearity
2. compute identity as FF^{-1}. Define delta function.
3. Parseval's Thm
4. convolution
5. discuss bandwidth and reciprocity. Eg. FT of 1 on a finite interval; Gaussian.

Lecture 23: FT of differential operators

1. general review of transform of operators under change of basis in R^n
2. compute transform of $F(d^n/dx^n)F^{-1}$ formally
3. diff. Eq. w/ const. coeff. transform to algebraic eq.
4. homogeneous solutions in the null space
5. find particular solution by F^{-1}

Lecture 24: Solution of inhomogeneous diff. eq. by Fourier methods

1. eg. harmonically driven, undamped oscillator (LC circuit)
2. briefly driven (sq. wave) H.O.: leave as integral
3. damped HO (LRC circuit), harmonically driven
4. damped HO, impulse ($\delta(t-t_0)$) driven: leave as integral
5. discuss the unevaluated integrals and motivate consideration in the complex plane

Lecture 25: Intro to complex variables

1. complex function of complex variables.
2. Cartesian and polar rep.
3. Exclude functions with branch cuts. eg $\sqrt{z}$, $\ln z$
4. define derivatives and differentiability.

Lecture 26: Differentiability, Cauchy–Riemann

1. eg. z^n, z^{-1}, $\bar{z}$, $\sqrt{z}$
2. derive C-R eq from differentiability
3. C-R eq. eg. $|z|^2$, $|z|$, z^2
4. C-R$\rightarrow$ Laplace's eq.
4. discuss connection to electrostatics and conservative forces

Lecture 27: Cauchy integral theorem

1. follow Greenberg, derive integral theorem for analytic functions (fudge Green's thm)
2. path independence
3. conservative forces again
4. derive residue for $(z-a)^n$ a la Greenberg p 1185
5. derive residue for simple poles

Lecture 28: Jordan's Lemma and integrals on the real line

1. Jordans lemma: stress this has nothing to do with poles, just asymptotic behavior.
2. closing integrals of the form $\int_{-\infty}^{\infty} \mathrm{d}x\, f(x)$
3. eg. $\int_{-\infty}^{\infty} \mathrm{d}x\, \sin(x)/(z-a)$; $\int_{-\infty}^{\infty} \mathrm{d}x\, \sin(x)/[(z-a)(z-b)]$
4. poles on the real line

Lecture 29: Evaluating the integral from Lecture 24

1. eg. damped impulse driven HO
2. undamped: confluent first order poles by limit of two first order poles
3. quick (3 line) proof of $\oint \mathrm{d}z f(z)/(z-a)^{n+1} = [2\pi \mathrm{i}/n!] f^{(n)}(a)$
4. show agreement between 2 and 3

Lecture 30: More examples

1. coupled LC circuits

Lecture 31: review

Lecture 32: review

Midterm II

Lecture 33: Multidimensional FT

1. properties: shift, scaling, Parseval'sThm.
2. transform of PDEs with constant coefficients

Lecture 34: Solution of Helmholtz eq. with point source

1. transform of the Helmholtz operator.
2. solution by IFT and evaluation of the integral by contour integration
3. discuss advanced and retarded sols
4. remove advanced sol by adding a little absorption to the background

Lecture 35: Differential vector calculus

1. grad div curl in Cartesian coords
2. def of div and curl by limit of integrals (grad for HW)
3. Laplacian and curl curl.

Lecture 36: More vector calc.

1. div, grad, curl in polar cyl. coords from integral def.
2. div, grad, curl in CPC by chain rule
3. Laplacian

Lecture 37: More vector calc.

1. more on CPC and examples.
2. note terms not proportional to derivatives. Why does this make sense?
3. eq. $\mathbf{v} = \hat{\theta}$

Lecture 38: Divergence theorem and Green's theorem

1. divergence thm as fundamental theorem of calc.
2. Green's identities as special case
3. application to electrostatics, Gauss' law.
4. Green's thm as fundamental thm of calc.

Lecture 39: Stokes theorem

1. expand smooth surfaces as locally flat and apply Green's thm.
2. deal with creases and folds.
3. beware singular fields, eg, mag. field due to filamentary current
4. closed surfaces. only edges matter.

Lecture 40: (optional) Introduction to Green's functions

1. general theory: point response for linear systems
2. Helmholtz eq.