Math. 231A

Review Problems

Solutions

1 (a) $\displaystyle\int \frac{2^{1-\sqrt{x}}}{\sqrt{x}}\,dx$

Put $u = 1-\sqrt{x}$, $du = -\frac{1}{2\sqrt{x}}\,dx$

$$\therefore \int \frac{2^{1-\sqrt{x}}}{\sqrt{x}}\,dx = \int 2^u(-2\,du)$$

$$= -2\cdot\frac{2^u}{\ln 2} + c$$

$$= -\frac{2^{2-\sqrt{x}}}{\ln 2} + c.$$

(b) $\displaystyle\int \sin^2 x \cos 3x\,dx$

$$= \int \frac{1}{2}(1-\cos 2x)\cos 3x\,dx$$

$$= \frac{1}{2}\int \cos 3x\,dx - \frac{1}{2}\int \cos 2x \cos 3x\,dx.$$

1 (b) (ctd).

Note that $\cos 2x \cos 3x = \frac{1}{2}(\cos 5x + \cos x)$.

Hence

$$\int \sin^2 x \cos 3x \, dx$$

$$= \frac{1}{2}\int \cos 3x \, dx - \frac{1}{4}\int (\cos 5x + \cos x) dx$$

$$= \frac{1}{6} \sin 3x - \frac{1}{20} \sin 5x - \frac{1}{4} \sin x + c.$$

(c) $\int x^2 (\ln x)^2 dx$ (Integrate by parts).

$$= \frac{1}{3} x^3 (\ln x)^2 - \int \frac{1}{3} x^3 \left(2 (\ln x) \frac{1}{x}\right) dx$$

$$= \frac{1}{3} x^3 (\ln x)^2 - \frac{2}{3} \int x^2 \ln x \, dx$$

$$= \frac{1}{3} x^3 (\ln x)^2 - \frac{2}{3} \left\{ \frac{1}{3} x^3 \ln x - \int \frac{1}{3} x^3 \cdot \frac{1}{x} dx \right\}$$

$$= \frac{1}{3} x^3 (\ln x)^2 - \frac{2}{9} x^3 \ln x + \frac{2}{27} x^3 + c.$$

2 (a) $\displaystyle\int_0^\infty \frac{x\,dx}{(1+x^2)^2}$

$$\int_0^t \frac{x}{(1+x^2)^2}\,dx = \left[-\frac{1}{2(1+x^2)}\right]_0^t \quad \left(\begin{array}{c}\text{put}\\ u=1+x^2\end{array}\right)$$

$$= \frac{1}{2}\left(1-\frac{1}{1+t^2}\right)$$

$\displaystyle\lim_{t\to\infty}\left(1-\frac{1}{1+t^2}\right) = 1.$ Hence

$$\int_0^\infty \frac{x}{(1+x^2)^2}\,dx = \frac{1}{2}.$$

(b) $$\int_0^3 \frac{dx}{x^2-1} = \int_0^1 \frac{dx}{x^2-1} + \int_1^3 \frac{dx}{x^2-1},$$

if both of these exist. Now

$$\int \frac{dx}{x^2-1} = \int \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx$$

$$= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + c.$$

$$\int_0^t \frac{dx}{x^2-1} = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|.$$

As $t\to 1-$, this $\to -\infty$. Integral diverges.

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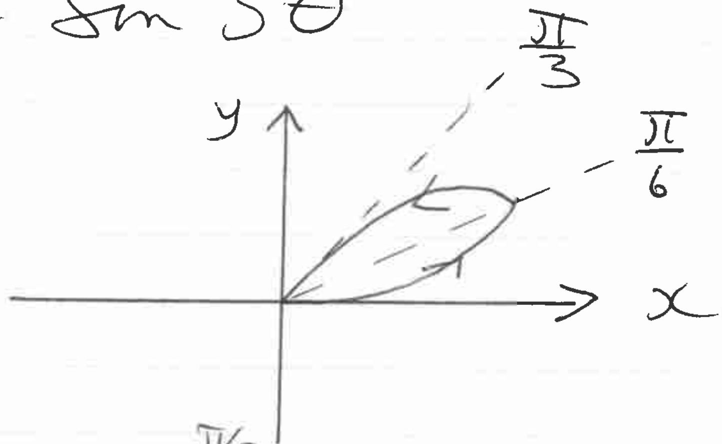
3. $r = \sin 3\theta$



Area $A = \int_0^{\pi/3} \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \int_0^{\pi/3} \sin^2 3\theta \, d\theta$

$\therefore A = \frac{1}{4} \int_0^{\pi/3} (1 - \cos 6\theta) \, d\theta$

$= \frac{1}{4} \left[\theta - \frac{\sin 6\theta}{6} \right]_0^{\pi/3}$

$= \frac{\pi}{12}.$

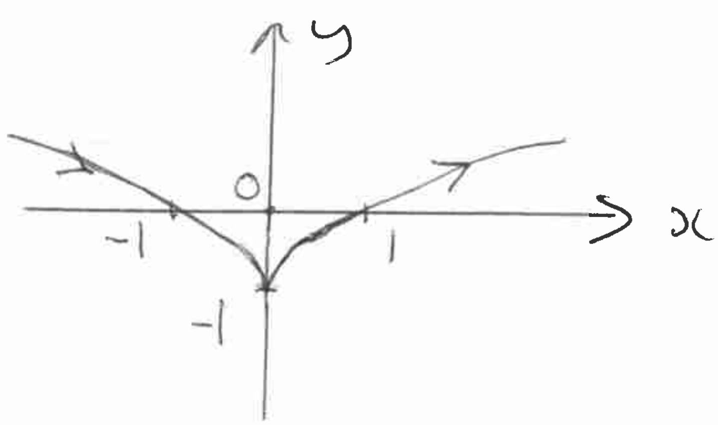
4. $\begin{cases} y = t^2 - 1 \\ x = t^3 \end{cases}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2} = \frac{2}{3t}.$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{2}{3t}\right)}{3t^2}$$

$$= -\frac{2}{9t^4}.$$

Note $\frac{dy}{dx}$ is never 0 and $\frac{dy}{dx}$ is infinite at $t = 0$.



t	$-\infty$	-1	0	1	$+\infty$
x	$-\infty$	-1	0	1	$+\infty$
y	$+\infty$	0	-1	0	$+\infty$

5. $\begin{cases} x = t^2 \\ y = t^3 \end{cases}$

$$s = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 \sqrt{4t^2 + 9t^4}\, dt$$

$$= \int_0^1 t\sqrt{4 + 9t^2}\, dt$$

Put $u = 4 + 9t^2$, $du = 18t\, dt$

$$\therefore s = \int_4^{13} \sqrt{u} \cdot \frac{1}{18}\, du$$

$$= \left[\frac{1}{27} u^{3/2}\right]_4^{13}$$

$$= \frac{1}{27}\left(13^{3/2} - 8\right).$$

6. Find the surface area of revolution about the x-axis of the curve $y = x^3$, $0 \leq x \leq 1$.

$$\text{Surface area } S = \int_0^1 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\therefore S = 2\pi \int_0^1 x^3 \sqrt{1+9x^4}\,dx$$

$$= 2\pi \left[\frac{1}{54}\left(1+9x^4\right)^{3/2}\right]_0^1 \quad \left(\begin{array}{c}\text{put}\\ u=1+9x^4\end{array}\right)$$

$$= \frac{\pi}{27}\left(10^{3/2}-1\right).$$

7. (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$

Use the Integral test.

$$\int_2^t \frac{1}{x\ln x}\,dx = \Big[\ln(\ln x)\Big]_2^t$$
$$= \ln(\ln t) - \ln(\ln 2)$$
$$\to \infty, \text{ as } t \to \infty.$$

So $\displaystyle\int_2^{\infty} \frac{dx}{x\ln x}$ diverges, and thus

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$ <u>diverges</u>

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$.

Note that $\ln(n) < n$, so

$\displaystyle\frac{1}{(\ln n)^2} \geq \frac{1}{n(\ln n)}$. Since

$\displaystyle\sum \frac{1}{n(\ln n)}$ diverges, <u>so does</u>

$\displaystyle\sum \frac{1}{(\ln n)^2}$, by the Comparison test.

7(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+1)}{(n-1)!} 2^n$

Use the Ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)(n+2)}{n!} 2^{n+1} \cdot \frac{(n-1)!}{(n+1)n 2^n}$$

$$= \frac{(n+2)\,2}{n^2} \to 0 \text{ as}$$

$n \to \infty$. Hence the series converges.

8. (a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{n\sqrt{n}-1}{n\sqrt{n}+1}\right)$.

Here $\lim\limits_{n\to\infty} |a_n| = 1$, so $\lim\limits_{n\to\infty} a_n \neq 0$.

By the nth term test the series diverges.

8(e). $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{1+n}$.

Note that $f(x) = \frac{\sqrt{x}}{1+x}$ is a decreasing function for $x \geq 1$ and $\lim_{x \to \infty} f(x) = 0$ (since $f'(x) = \frac{(1-x)}{2\sqrt{x}\,(1+x)^2} < 0$). Therefore the series converges by the Alternating Series Test.

But $\frac{\sqrt{n}}{1+n} > \frac{\sqrt{n}}{n+n} = \frac{1}{2\sqrt{n}}$

and $\sum \frac{1}{\sqrt{n}}$ diverges. Hence

$\sum \frac{\sqrt{n}}{1+n}$ diverges, and the given series <u>converges conditionally</u>.

9. $f(x) = \tan x$, $f(0) = 0$.

$f'(x) = \sec^2 x$, $f'(0) = 1$.

$f''(x) = 2\sec^2 x \tan x$, $f''(0) = 0$.

$f^{(3)}(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$

$\therefore f^{(3)}(0) = 2$.

$f^{(4)}(x) = 8\sec^2 x \tan^3 x + 8\sec^4 x \tan x$.

$\therefore f^{(4)}(0) = 0$. Finally,

$f^{(5)}(0) = 8$.

The series is therefore

$$\frac{1}{1!}x + \frac{2}{3!}x^3 + \frac{8}{5!}x^5 + \cdots$$

i.e.

$$x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \cdots$$

10. $f(x) = \int_0^x \frac{e^t - 1}{t}\, dt$

$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} \quad \frac{t^n}{n!} + \cdots$$

$$\therefore \frac{e^t - 1}{t} = \frac{1}{1!} + \frac{t}{2!} + \cdots + \frac{t^{n-1}}{n!} + \cdots$$

$$\int_0^x \frac{e^t - 1}{t}\, dt = \frac{x}{1!} + \frac{x^2}{2(2!)} + \frac{x^3}{3(3!)} + \cdots$$

$$\cdots + \frac{x^n}{n(n!)} + \cdots$$

Valid for all x: the radius of convergence is

$$\lim_{n \to \infty} \frac{\frac{1}{n(n!)}}{\frac{1}{(n+1)(n+1)!}} = \lim_{n \to \infty} \frac{(n+1)^2}{n} = \infty.$$