

## Math 415 Exam I: June 27, 2011; Solutions

1. Find the inverse of  $A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}$ .

By Gaussian–Jordan elimination, we have

$$\begin{bmatrix} 2 & 1 & 4 & 1 & 0 & 0 \\ 3 & 2 & 5 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -7 & 5 & 3 \\ 0 & 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{bmatrix}.$$

Thus,  $A^{-1}$  is the last 3 columns of second matrix.

2. Consider the following linear system:

$$\begin{aligned} 2x_1 + x_2 + 4x_3 + x_4 &= 0 \\ 3x_1 + 2x_2 + 5x_3 + x_5 &= 0 \\ -x_2 + x_3 + &= 1. \end{aligned}$$

- (i) (5 points) Which variables, if any, are free variables?

The augmented matrix of this linear system is the left hand side  $3 \times 6$  matrix in the first problem. Thus, we see that  $x_4$  and  $x_5$  are free variables.

- (ii) (20 points) Find all solutions (if any).

The linear system corresponding to the reduced row echelon form in the first problem is equivalent to the original system (i.e., it has the same solutions). But the new system is

$$\begin{aligned} x_1 &= 7x_4 - 5x_5 + 3 \\ x_2 &= -3x_4 + 2x_5 - 2 \\ x_3 &= -3x_4 + 2x_5 - 1. \end{aligned}$$

If we set  $x_4 = a$  and  $x_5 = b$ , then a 5-tuple  $\mathbf{s}$  is a solution if

$$\mathbf{s} = (7a - 5b + 3, -3a + 2b - 2, -3a + 2b - 1, a, b).$$

- 3(i) (10 points) Prove or find a concrete counterexample. If  $A$  and  $B$  are  $n \times n$  matrices, then  $(AB)^2 = A^2B^2$ .

There are many examples; here is one. Set  $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ . Then  $B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ , so that  $A^2B^2 = A^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ . On the other hand,  $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  and  $(AB)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ , which is not the zero matrix.

(ii) (5 points) Define what it means to say that an  $m \times n$  matrix  $A$  is in **row echelon form**.

$A$  satisfies the following three conditions.

1. all zero rows, if any, are at the bottom.
2. the leading entry of every nonzero row is 1.
3. the column containing the leading entry of a nonzero row is to the right of the column containing the leading entry of any higher row.

(iii) (5 points) Define what it means to say that an  $m \times n$  matrix  $A$  is in **reduced row echelon form**.

$A$  is in row echelon form and every leading entry is the only nonzero entry in its column.

(iv) (5 points) Say whether each of the following matrices is in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The first two matrices are reduced row echelon form; the other three are not (the third and fifth are not even in row echelon form).

4. Recall that a square matrix  $A$  is **symmetric** if  $A^\top = A$ .

(i) (10 points) Give an example of two symmetric  $n \times n$  matrices  $A$  and  $B$  such that  $AB$  is not symmetric.

There are many examples; here is one. Let  $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ . Both  $A$  and  $B$  are symmetric, but  $AB = \begin{bmatrix} * & 7 \\ 4 & * \end{bmatrix}$  is not symmetric.

(ii) (5 points) If  $A$  and  $B$  are symmetric  $n \times n$  matrices that commute ( $AB = BA$ ), prove that  $AB$  is symmetric.

$$(AB)^\top = B^\top A^\top = BA = AB.$$

(iii) (10 points) If  $A$  is symmetric, prove that  $A^m$  is symmetric for all  $m \geq 1$ . [You may assume (the easily proved fact) that  $A$  commutes with each of its powers.]

The proof is by induction on  $m \geq 1$ . The base step is obvious. For the inductive step, we must prove that  $A^{m+1}$  is symmetric if  $A^m$  is. But  $A^{m+1} = A^m A$ . Now both  $A^m$  and  $A$  are symmetric (the former by the inductive hypothesis). As both  $A^m$  and  $A$  commute, part (ii) says that their product  $A^{m+1}$  is symmetric.