

Math 415 Exam II Solutions: July 15, 2011

1(i). (5 points) Show that the subset $S = \{A \in \mathbb{R}^{3 \times 3} : \det(A) = 0\}$ is not a subspace of $\mathbb{R}^{3 \times 3}$.

S is not closed under addition. For example, let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Now $\det(A) = 0 = \det(B)$, but $A + B = I$, the identity matrix, and $\det(A + B) = 1$.

(ii). (5 points) Show that the subset $U = \{A \in \mathbb{R}^{4 \times 4} : A \text{ is symmetric}\}$ is a subspace of $\mathbb{R}^{4 \times 4}$.

The zero matrix is symmetric. If

$$A = \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix} \text{ and } A' = \begin{bmatrix} a' & b' & c' & d' \\ b' & e' & f' & g' \\ c' & f' & h' & i' \\ d' & g' & i' & j' \end{bmatrix},$$

then both

$$A + A' = \begin{bmatrix} a + a' & b + b' & c + c' & d + d' \\ b + b' & e + e' & f + f' & g + g' \\ c + c' & f + f' & h + h' & i + i' \\ d + d' & g + g' & i + i' & j + j' \end{bmatrix} \text{ and } \alpha A = \begin{bmatrix} \alpha a & \alpha b & \alpha c & \alpha d \\ \alpha b & \alpha e & \alpha f & \alpha g \\ \alpha c & \alpha f & \alpha h & \alpha i \\ \alpha d & \alpha g & \alpha i & \alpha j \end{bmatrix}$$

are symmetric. Hence, U is a subspace.

(iii). (15 points) Find the dimension of the subspace U in part (ii).

$\dim(U) = 10$. For every i, j with $1 \leq i \leq 4$ and $1 \leq j \leq 4$, let E_{ij} be the 4×4 matrix having 1 as its ij entry and all other entries 0. Then a basis for U consists of the matrices $E_{11}, E_{22}, E_{33}, E_{44}$ and the six matrices $E_{ij} + E_{ji}$ for all $i \neq j$. These matrices span U : the matrix A in part (ii) is equal to

$$aE_{11} + b(E_{12} + E_{21}) + c(E_{13} + E_{31}) + \text{etc.}$$

These matrices are linearly independent, for a linear combination of them being the zero matrix says each of the entries a, b, c , etc is zero.

2. Let $A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ -1 & 0 & 3 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 4 & 0 & 5 \end{bmatrix}$.

(i). (10 points) Find $\det(A)$ with Laplace expansion down the second column.

Expanding down the second column, we have

$$\det(A) = (-1) \det \begin{bmatrix} -1 & 3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} + 2 \det \begin{bmatrix} 0 & -2 & 1 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

(remember the checkerboard pattern of + and -). The first 3×3 determinant is 6 (the matrix is diagonal) while the second determinant is 0 (the matrix has two identical rows), and so $\det(A) = -6$.

(ii). (10 points) Find $\text{adj}(B)$.

$\text{adj}(B) = [B_{ij}]^\top$, the transpose of the matrix whose entries are the cofactors $B_{ij} = (-1)^{i+j} \det(M_{ij})$, where M_{ij} is the 2×2 submatrix obtained from B by deleting row i and column j .

$$\text{adj}(B) = \begin{bmatrix} 15 & 0 & -6 \\ 0 & -3 & 0 \\ -12 & 0 & 3 \end{bmatrix}.$$

(ii). (5 points) Find B^{-1} .

It is easy to see that $\det(B) = -9$: Laplace expansion across row 2 gives

$$\det(B) = 3 \det \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = 3 \times (5 - 8) = -9.$$

Hence,

$$B^{-1} = \frac{1}{-9} \begin{bmatrix} 15 & 0 & -6 \\ 0 & -3 & 0 \\ -12 & 0 & 3 \end{bmatrix}.$$

3. Let $u_1 = (4, 1, 0)^\top$, $u_2 = (3, 1, 0)^\top$, $u_3 = (0, 0, -1)^\top$ in \mathbb{R}^3 .

(i). (10 points) Show that u_1, u_2, u_3 is a basis of \mathbb{R}^3 .

The matrix $U = \begin{bmatrix} 4 & 3 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ whose columns are the u 's, is nonsingular, because its determinant -1 is nonzero.

(ii) (15 points). Write $x = (0, 4, 3)^\top$ as a linear combination of u_1, u_2, u_3 .

If $c = (c_1, c_2, c_3)^\top$ are the coefficients we seek, then $Uc = x$, and so $c = U^{-1}x$. Now

$$U^{-1} = \begin{bmatrix} 1 & -3 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

so that $c = (-12, 16, -3)^\top$; that is,

$$x = -12u_1 + 16u_2 - 3u_3.$$

4(i). (10 points) Let A be a 14×27 matrix. If the solution space of $Ax = 0$ has dimension 18, find $\text{rank}(A)$, and explain why your answer is correct.

We know that the solution space of the linear system $Ax = 0$ is the same as the null space $N(A)$. We also know that $\dim(N(A)) = n - r$, where A is an $m \times n$ matrix of rank r . Here, $n = 27$ and $n - r = 18$, so that $27 - r = 18$. Therefore, $r = 9$.

(ii). (15 points) Let U be a subspace of \mathbb{R}^n . Prove that $\dim(U) \leq n$. You may assume that U is finite-dimensional.

Since U is finite-dimensional, it has a basis, say $X = u_1, \dots, u_m$ (actually, it is not difficult to prove that U is finite-dimensional; see below). Now X is linearly independent. Suppose that $m > n$. Since $\dim(\mathbb{R}^n) = n$, we are in trouble, for X can be extended to be part of some basis of \mathbb{R}^n , and this basis is too big: after all, we know that any two bases of a vector space have the same number of elements.

Here is a proof (that I allowed you to assume) that U must be finite-dimensional; that is, U has a finite spanning set. If $U = \{0\}$, we are done. Otherwise, there is $u_1 \in U$ with $u_1 \neq 0$. The (short) list u_1 is linearly independent. If it spans U , we are done. Otherwise, there is $u_2 \in U$ which is not in $\text{Span}(u_1)$. We saw in class that the list u_1, u_2 is linearly independent. If $U = \text{Span}(u_1, u_2)$, we're done. Keep doing this. Either this procedure stops after, say, m steps, in which case $U = \text{Span}(u_1, \dots, u_m)$ and we're done, or the procedure never stops. In particular, the procedure can go on $n + 1$ steps, giving a linearly independent list u_1, \dots, u_{n+1} in U , hence in \mathbb{R}^n . As in the above proof, this would lead to a basis of \mathbb{R}^n with too many elements.