Math 415 Exam III Solutions: August 1, 2011

1. Let A be an $n \times n$ matrix.

(**i**). (**5 points**) Using the fact that the constant term a_0 of a polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is equal to $f(0)$, prove that the constant term of the characteristic polynomial is $\det(A)$.

By definition, $p(\lambda) = \det(A - \lambda I)$. So, using the fact above, the constant term is $p(0) = \det(A - 0I) = \det(A)$.

(**ii**). (**10 points**) If A is nonsingular, show that all its eigenvalues are nonzero.

If $\lambda = 0$ is an eigenvalue, then there is a nonzero vector v with $Av = 0$; that is, v is a nontrival solution of the homogeneous system $Ax = 0$. Hence, A is singular, a contradiction. There are other proofs.

(**iii**). (**10 points**) If $A^k = 0$ for some $k \geq 1$, prove that all the eigenvalues of A are 0.

If $Av = \lambda v$, then repeating (or using induction) gives $A^k v = \lambda^k v$ for all $k \geq 1$. Now $A^k = 0$, so that $\lambda^k v = 0$. Since v is an eigenvector, $v \neq 0$; hence, $\lambda^k = 0$ and $\lambda = 0$.

2. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \\ -1 & 0 & 3 \end{bmatrix}$.

(**i**). (**15 points**) Find a basis for the column space of A.

The column space of A is the row space $R(A^\top)$ of

$$A^\top = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 3 & 0 \\ 2 & 1 & 4 & 3 \end{bmatrix}.$$

The reduced row echelon form for $A^\top$ is

$$E = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

A basis for the column space of A are the rows of E (and so its dimension is 3).

(**ii**). (**10 points**) What is the dimension of the orthogonal complement of the column space of A? You must support your answer; if you use a theorem from class, you must state that theorem correctly.

The relevant theorem says that if S is a subspace of $\mathbb{R}^n$, then $n = \dim(S) + \dim(S^\perp)$. Thus, $\dim(R(A^\top)^\perp) = 4 - 3 = 1$.

Another solution uses the theorem: $(R(A^\top)^\perp) = N(A)$. Thus, $\dim(N(A)) = n - r$. But $r = \mathrm{rank}(A) = 3$, and $4 - 3 = 1$.

3. Let $u_1 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})^\top, u_2 = (\frac{4}{\sqrt{42}}, \frac{-5}{\sqrt{42}}, \frac{1}{\sqrt{42}})^\top, u_3 = (\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-3}{\sqrt{14}})^\top$.

(**i**). (**6 points**) Show that u_1, u_2, u_3 is an orthonormal basis of $\mathbb{R}^3$.

$$\begin{aligned}
\|u_1\|^2 &= \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3} = 1. \\
\|u_2\|^2 &= \tfrac{16}{42} + \tfrac{25}{42} + \tfrac{1}{42} = 1. \\
\|u_3\|^2 &= \tfrac{4}{14} + \tfrac{1}{14} + \tfrac{9}{14} = 1. \\
(u_1, u_2) &= \tfrac{1}{126}(4 - 5 + 1) = 0. \\
(u_1, u_3) &= \tfrac{1}{42}(2 + 1 - 3) = 0. \\
(u_1, u_2) &= \tfrac{1}{42 \cdot 14}(8 - 5 - 3) = 0.
\end{aligned}$$

(**ii**) (**9 points**). Write $x = (0, 4, 3)^\top$ as a linear combination of u_1, u_2, u_3.

Let $x = c_1 u_1 + c_2 u_2 + c_3 u_3$. Since u_1, u_2, u_3 is an orthonormal basis, each coefficient is a scalar product: $c_i = (x, u_i)$. Thus,

$$c_1 = \frac{7}{\sqrt{3}}, \qquad c_2 = \frac{-17}{\sqrt{42}}, \qquad c_3 = \frac{-5}{\sqrt{14}}.$$

(**iii**) (**10 points**). Find A^{-1} if $A = \begin{bmatrix} 0&1&0&0&0 \\ 0&0&1&0&0 \\ 1&0&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&1&0 \end{bmatrix}$.

Since A is a permutation matrix, it is an orthogonal matrix, and so its inverse is its transpose.

4. Let $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$.

(**i**) (**10 points**) Find linearly independent eigenvectors v_1, v_2 of A.

The characteristic polynomial of A is $p(\lambda) = \lambda^2 - 6\lambda + 5$, and so its eigenvalues are $1, 5$.

An eigenvector belonging to 1 is found by solving the homogeneous system $(A - I)x = 0$. Now $A - I = \left[\begin{smallmatrix} 3 & 3 \\ 1 & 1 \end{smallmatrix}\right]$, and it is easy to see that $v_1 = (-1, 1)^\top$ is a solution. Also, $A - 5I = \left[\begin{smallmatrix} -1 & 3 \\ 1 & -3 \end{smallmatrix}\right]$, and it is easy to see that $v_2 = (3, 1)^\top$ is a solution. It is plain v_1, v_2 is a linearly independent list.

(**ii**) (**10 points**) Find a matrix P and a diagonal matrix D such that $D = PAP^{-1}$.

Let $D = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 5 \end{smallmatrix}\right]$ be the diagonal matrix with the same eigenvalues as A, and let P be the transition matrix from v_1, v_s to the standard basis:

$$P = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$$

Now

$$P^{-1} = -\tfrac{1}{4}\begin{bmatrix} 1 & -3 \\ -1 & -1 \end{bmatrix},$$

and $PAP^{-1} = D$.

(iii) (**5 points**) Find A^{100}.

We have $A = P^{-1}DP$, so that

$$A^{100} = (P^{-1}DP)^{100} = P^{-1}D^{100}P,$$

and

$$D^{100} = \begin{bmatrix} 1 & 0 \\ 0 & 5^{100} \end{bmatrix}.$$