# Math 415 Final Exam Solutions: August 5, 2011

**1**. Define each of the first five terms.

(**i**) (**4 points**) The list $v_1, \ldots, v_n$ of vectors is linearly independent.

Whenever there are scalars $c_1, \ldots, c_n$ with $c_1 v_1 + \cdots + c_n v_n = 0$, then all $c_i = 0$.

(**ii**) (**4 points**) Dimension of a finite-dimensional vector space $V$.

The dimension of $V$ is the number of elements in a basis of $V$.

(**iii**) (**4 points**) Orthogonal complement $S^\perp$ of a subspace $S$ of a vector space $V$.

$$S^\perp = \{v \in V : (v, s) = 0 \text{ for all } s \in S\}.$$

(**iv**) (**4 points**) A Hermitian matrix $A = [a_{jk}]$. (You must explain your notation.)

$A^H = A$. Here, $A^H = [\overline{a}_{ij}]^\top$.

(**v**) (**4 points**) An eigenvalue of a linear operator $L \colon V \to V$.

A number $\lambda$ for which there exists a nonzero vector $v$ with $Av = \lambda v$.

(**vi**) (**5 points**) What are the columns of the transition matrix $P$ from a basis $X$ to a basis $Y$?

If $X = x_1, \ldots, x_n$, then the $j$th column of $P$ consists of the coordinates of $x_j$ with respect to $Y$.

**2**. Consider the homogeneous linear system

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 \qquad &= 0 \\ 2x_1 + 2x_2 + 3x_3 + x_4 \qquad &= 0 \\ x_1 - x_2 + 2x_3 + 3x_4 + x_5 &= 0 \end{aligned}$$

**i**) (**10 points**) Find a basis of the row space of this system.

The row reduced echelon form for the coefficient matrix is

$$E = \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{2} \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix}.$$

A basis of the row space consists of the rows of $E$.

**ii**) (**15 points**) Find a basis of the solution space of this system.

Using $E$, we see that the free variables are $x_4$ and $x_5$, and

$$\begin{aligned} x_1 &= \tfrac{3}{2}x_4 - \tfrac{1}{2}x_5 \\ x_2 &= \tfrac{1}{3}x_4 + \tfrac{1}{2}x_5 \\ x_3 &= -3x_4 \end{aligned}$$

Thus, a basis for the solution space is

$$(\tfrac{3}{2}, \tfrac{1}{3}, -3)^\top, (-\tfrac{1}{2}, \tfrac{1}{2}, 0)^\top.$$

**3**. Let $A$ be an $n \times n$ matrix, where $n \geq 2$.

(**i**) (**10 points**) If $A$ is singular, prove that $\operatorname{adj}(A)$ is singular.

We know that $A \operatorname{adj}(A) = \det(A)I$. Since $A$ is singular, $\det(A) = 0$, and so $A \operatorname{adj}(A) = 0$. If $A = 0$, then $\operatorname{adj}(A) = 0$; if $A \neq 0$, then $\operatorname{adj}(A)$ is singular: otherwise, multiply on the right by $\operatorname{adj}(A)^{-1}$ and get $A = 0$.

(**ii**) (**15 points**) If $A$ is nonsingular, find $\det(\operatorname{adj} A)$; justify your answer.

Since $A \operatorname{adj}(A) = \det(A)I$, we have

$$\det(A \operatorname{adj}(A)) = \det(A)\det(\operatorname{adj}(A)) = \det(A)^n.$$

Therefore, $\det(\operatorname{adj}(A)) = \det(A)^{n-1}$.

**4**. Let $V$ and $W$ be vector spaces, let $U = u_1, \ldots, u_n$ be a basis of $V$, and let $w_1, \ldots, w_n$ be a list of (not necessarily distinct) vectors in $W$. Prove that there is a unique linear transformation $L\colon V \to W$ with $L(u_i) = w_i$ for all $i$. You must show that the $L$ you construct is single-valued.

Each $v \in V$ has coordinates wrt $U$: there are unique scalars $c_1, \ldots, c_n$ with $v = c_1u_1 + \cdots + c_nu_n$. Define $L$ by

$$L(v) = c_1w_1 + \cdots + c_nw_n.$$

Note that uniqueness of the coordinates shows that $L$ is a single-valued function. Obviously, $L(u_i) = w_i$ for all $i$, and so it remains to prove that $L$ is a linear transformation.

If $v' = c_1'u_1 + \cdots + c_n'u_n$, then $v + v' = (c_1 + c_1')u_1 + \cdots + (c_n + c_n')u_n$, and so $L(v + v') = (c_1 + c_1')w_1 + \cdots + (c_n + c_n')w_n$. On the other hand, $L(v) + L(v') = [c_1w_1 + \cdots + c_nw_n] + [c_1'w_1 + \cdots + c_n'w_n] = (c_1 + c_1')u_1 + \cdots + (c_n + c_n')u_n$.

Finally, if $\alpha$ is a scalar, then

$$\begin{aligned} L(\alpha v) &= L(\alpha c_1u_1 + \cdots + \alpha c_nu_n) \\ &= \alpha c_1w_1 + \cdots + \alpha c_nw_n \\ &= \alpha(c_1w_1 + \cdots + c_nw_n) \\ &= \alpha L(v). \end{aligned}$$

To prove uniqueness, suppose that $T\colon V \to W$ is a linear transformation with $T(u_i) = w_i$ for all $i$. Since $T$ preserves linear combinations, we have

$$\begin{aligned} T(v) &= T(c_1u_1 + \cdots + c_nu_n) \\ &= c_1T(u_1) + \cdots + c_nT(u_n) \\ &= c_1w_1 + \cdots + c_nw_n \\ &= L(v). \end{aligned}$$

Therefore, $T = L$.

**5**. Let $S$ be the set of all $4 \times 4$ skew-symmetric matrices.

(**i**) (**10 points**) Prove that $S$ is a subspace of $\mathbb{R}^{4\times 4}$.

Recall that $A$ is skew-symmetric if $A^\top = -A$. Clearly, the zero matrix is skew-symmetric: $0 \in S$. If $A, B \in S$, then $A^\top = -A$ and $B^\top = -B$, so that $(A+B)^\top = A^\top + B^\top = -A - B = -(A+B)$; thus, $A + B \in S$. Finally, if $\alpha$ is a scalar and $A \in S$, then $(\alpha A)^\top = \alpha(A^\top) = \alpha(-A) = -(\alpha A)$, and so $\alpha A \in S$. Therefore, $S$ is a subspace.

(**ii**) (**15 points**) Find $\dim(S)$; justify your answer.

Let $E_{ij}$ be the $4 \times 4$ matrix having 1 in the $ij$ spot and all other entries 0. Now skew-symmetric matrices $A = [a_{ij}]$ must have 0's on the diagonal: since $a_{ji} = -a_{ij}$, we have $a_{ii} = -a_{ii}$. Therefore,

$$A = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix},$$

and a basis for $S$ consists of all $E_{ij} - E_{ji}$ for all $i < j$. Hence, $\dim(S) = 6$.

**6**. (**25 points**) Let $p$ be a solution of the system $Ax = b$, where $b \neq 0$. Prove that every solution has a unique expression of the form $u + p$, where $u$ is a solution of the homogeneous system $Ax = 0$.

Let $S$ be all the solutions of $Ax = b$, and let $Z$ be the set of all $u + p$ where $Au = 0$.

We claim $S \subseteq Z$. If $s \in S$, then $As = b$. Hence, $A(s - p) = As - Ap = b - b = 0$; that is, $s - p$ is a solution of $Ax = 0$, and so $s = (s - p) + p \in Z$.

For the reverse inclusion $Z \subseteq S$, take $u + p \in Z$. Then $Au = 0$ and $Ap = b$, so that $A(u + p) = Au + Ap = 0 + b = b$; that is, $u + p \in S$.

For uniqueness, suppose that $u + p = u' + p$, where $Au = 0 = Au'$. Just cancel $p$ to obtain $u = u'$.

**7**. Find the inverse of $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

You can either use $\operatorname{adj}(A)$ or Gaussian elimination. Either way,

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 1 & -3 & -1 \\ 1 & 1 & -1 \\ -1 & 3 & 5 \end{bmatrix}.$$

**8**. Let $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$.

(**i**) (**10 points**) Prove that there is no matrix $P$ with real entries such that $P^{-1}AP$ is diagonal.

The characteristic polynomial of $A$ is $p(\lambda) = \lambda^2 - \sqrt{3}\lambda + 1$, and the quadratic formula gives the eigenvalues to be $\frac{1}{2}(\sqrt{3} \pm i)$. Now if $PAP^{-1} = D$, where $D$ is diagonal, then $D$ has real entries and its diagonal entries would have to be the eigenvalues of $A$, for similar matrices have the same eigenvalues. But the eigenvalues of $A$ are not real.

(**ii**) (**15 points** Find a matrix $Q$ with complex entries such that $Q^{-1}AQ$ is diagonal.

If $Y = y_1, y_2$ is a basis of eigenvalues of $\mathbb{R}^2$, then the matrix of $A$ wrt to $Y$ is a diagonal matrix with the eigenvalues of $A$ on the diagonal. Thus, a matrix $Q$ is a transition matrix from $Y$ to the standard basis.

An eigenvector of $A$ belonging to $\sqrt{3} + i$ is $(i, 1)^\top$, and an eigenvector of $A$ belonging to $\sqrt{3} - i$ is $(-i, 1)^\top$. Thus,

$$Q = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$$

is such a matrix (since eigenvectors are determined up to nonzero scalar multiples, $Q$ is not unique: we can multiply the first column by any nonzero complex number $\alpha$ and the second column by any nonzero complex number $\beta$).