Homework I: June 16, 2011

1. Find a formula for $1+\sum_{j=1}^{n} j!j$; use induction to prove that your formula is correct.

(i) A list of the sums for $n = 1, 2, 3, 4, 5$ is $2, 6, 24, 120, 720$. These are factorials; better, they are $2!, 3!, 4!, 5!, 6!$. If we write $S(n) = 1 + \sum_{j=1}^{n} j!j$, then our guess is $S(n) = (n + 1)!$.

(ii) We have already checked the base step $n = 1$, for $S(1) = 2 = 2!$. For the inductive step, we must prove that $S(n)$ implies

$$S(n+1) = 1 + \sum_{j=1}^{n+1} j!j = (n+2)!.$$

Rewrite the middle expression:

$$\left[\sum_{j=1}^{n} j!j\right] + (n+1)!(n+1).$$

The inductive hypothesis says that the bracketed term is $(n + 1)!$, and so

$$\begin{aligned} S(n+1) &= (n+1)! + (n+1)!(n+1) \\ &= (n+1)!(n+2) \\ &= (n+2)!. \end{aligned}$$

By induction, $S(n) = (n + 1)!$ for every $n \geq 1$.

Page10, **Ex.6**. Solve each of the following systems.

(d)

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 3 \\ -x_1 + 2x_2 + 3x_3 &= 7 \end{aligned}$$

We use Gaussian elimination to put the augmented matrix in row echelon form:

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & -1 & 1 & 3 \\ -1 & 2 & 3 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -5 & 3 & 1 \\ 0 & 4 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -5 & 3 & 1 \\ 0 & 0 & \frac{22}{5} & \frac{44}{5} \end{bmatrix}$$

The 3×3 coefficient matrix is in strict triangular form, and there are two consequences. First, there is a unique solution. Second, we can find this solution by back substitution (work from the bottom up). The solution is $(1, 1, 2)$.

(h)

$$\begin{aligned} x_2 + x_3 + x_4 &= 0 \\ 3x_1 \quad + 3x_3 - 4x_4 &= 7 \\ x_1 + x_2 + x_3 + 2x_4 &= 6 \\ 2x_1 + 3x_2 + x_3 + 3x_4 &= 6 \end{aligned}$$

The augmented matrix of this system is

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 3 & -4 & 7 \\ 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & 1 & 3 & 6 \end{bmatrix}$$

After doing elementary row operations, I obtained the following matrix which is almost in row echelon form:

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 6 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & -7 & -11 \\ 0 & 0 & 0 & -\frac{23}{3} & -\frac{40}{3} \end{bmatrix}$$

This matrix is not in echelon form because the leading entries in rows 3 and 4 are not equal to 1. In fact, the 4×4 coefficient matrix is in strict triangular form, and so we can draw the same conclusions as in part (d): there is a unique solution, and we can compute it by back substitution. The solution is

$$\left(\tfrac{178}{23}, \tfrac{573}{23}, -\tfrac{533}{23}, -\tfrac{40}{23}\right).$$