Homework 10: July 28, 2011

Page 258, **Ex**. **15** Let Q be an orthogonal matrix and let $d = \det(Q)$. Show that $d = \pm 1$.

Since $Q^\top Q = I$, we have $\det(Q^\top Q) = \det(I) = 1$, and so

$$\det(Q^\top Q) = \det(Q^\top)\det(Q) = \det(I) = 1.$$

But $\det(Q^\top) = \det(Q) = d$, so that $d^2 = 1$ and $d = \pm 1$.

Page 258, **Ex**. **18** Show that if P is a symmetric permutation matrix, then $P^{2k} = I$ and $P^{2k+1} = A$.

Since P is symmetric, we have $P^\top = P$; since permutation matrices are orthogonal, we have $P^\top = P^{-1}$. Thus, $P = P^{-1}$, and so $P^2 = I$. It follows that every even power of P is the identity matrix (for $P^{2k} = (P^2)^k = I^k = I$). Hence, $P^{2k+1} = P^{2k}P = IP = P$.

Page 294, **Ex**. **2** Show that the eigenvalues of a triangular matrix are the diagonal entries of the matrix.

If A is a triangular matrix, then so is $A - \lambda I$, for we are merely replacing the diagonal entries a_{ii} of A by $a_{ii} - \lambda$ and leaving all the other entries unchanged. Since $A - \lambda I$ is triangular, its determinant is just the product of its diagonal entries:

$$p(\lambda) = \det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda)\cdots(a_{nn} - \lambda).$$

Thus, the roots of $p(\lambda)$ are just the diagonal entries.

Two comments.

First, it is quite possible that $p(\lambda)$ has repeated roots.

Second, most people define the characteristic polynomial as $\det(\lambda I - A)$ instead of $\det(A - \lambda I)$. There is only a difference in sign. We know that multiplying one row of a matrix by a scalar α multiplies the determinant by α. Multiplying a second row by the same scalar α multiplies the original determinant by α^2. Thus, if B is an $n \times n$ matrix, then $\det(\alpha B) = \alpha^n \det(B)$. In particular, if $\alpha = -1$, then $\det(A - \lambda I) = (-1)^n \det(\lambda I - A)$. I said in class that the characteristic polynomial of an $n \times n$ matrix is a monic polynomial of degree n. However, with the book's definition, the coefficient of λ^n is $(-1)^n$. Of course, both versions of the characteristic polynomial have the same roots.

Page 295, **Ex**. **26** Let $B = S^{-1}AS$ and let x be an eigenvector of B belonging to an eigenvalue λ. Show that Sx is an eigenvector of A belonging to λ.

We are told that $Bx = \lambda x$, and we are to show that $ASx = \lambda Sx$. But this is easy. Since $B = S^{-1}AS$, we have $\lambda x = Bx = S^{-1}ASx$. Therefore, $S\lambda x = ASx$;

since $S\lambda x = \lambda Sx$, we are almost done. Why aren't we done? Eigenvectors have to be nonzero! But $x \neq 0$, since it is an eigenvector (of B), and $Sx \neq 0$, because S is nonsingular (so $\ker S = \{0\}$.

Notice what this says: Similar matrices have the same eigenvalues.