Homework III: June 23, 2011

Page 43, **8(c)**. If $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ 0 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, verify that

$$A(B + C) = AB + AC.$$

We calculate. First, left hand side.

$$A(B + C) = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \left(\begin{bmatrix} -2 & 1 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \right) = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 24 \\ 7 & 17 \end{bmatrix}$$

Now the right hand side.

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 18 \\ -2 & 13 \end{bmatrix} + \begin{bmatrix} 14 & 6 \\ 9 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 24 \\ 7 & 17 \end{bmatrix}.$$

Page 44, **15**. A matrix A is said to be *skew symmetric* if $A^\top = -A$. Show that if a matrix is skew symmetric, then its diagonal entries must all be 0.

If $A = [a_{ij}]$, then $A^\top = [a_{ji}]$ for all i, j. Hence, if A is skew symmetric, then $a_{ji} = -a_{ij}$ for all i, j. In particular, $a_{ii} = -a_{ii}$ and $a_{ii} = 0$.

Page 56, **8**. Let

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$$

Compute A^2 and A^3. Just using the definition of matrix multiplication, along with some patience, we obtain $A^2 = I$ and $A^3 = A$.

Here is a less computational way to do this. If J is the 4×4 matrix all of whose entries are 1, then $J^2 = 4J$ (why?); hence, $(\frac{1}{2}J)^2 = \frac{1}{4}J^2 = \frac{1}{4}4J = J$.

Now $A = \frac{1}{2}(2I - J)$, where I is the 4×4 identity matrix. But $2I$ commutes with J (in fact, for any scalar t, the matrix tI commutes with every matrix), and so we can use the Binomial Theorem:

$$A^2 = \tfrac{1}{4}(4I - 4J + J^2) = \tfrac{1}{4}(4J - 4J + 4I) = I.$$

And it is now easy to compute $A^3 = AA^2 = AI = A$.

The last part of the problem can be done by induction. We prove that $A^{2n+1} = A$ by induction on $n \geq 0$. The base step $n = 0$ says that $A^1 = A$; sure it is. For the inductive step, the inductive hypothesis $A^{2n+1} = A$ gives

$$A^{2(n+1)+1} = A^{(2n+1)+2} = AA^2 = A^3 = A,$$

for we have already verified that $A^3 = A$.

We prove that $A^{2n} = I$ by induction on $n \geq 0$. The base step $n = 0$ says that $A^0 = I$; this is the definition of exponent 0: $M^0 = I$ for every matrix M (if you don't like this, start your induction at $n = 1$, so that the base step is $A^2 = I$, which we have already verified). For the inductive step, the inductive hypothesis $A^{2n} = I$ gives

$$A^{2(n+1)} = A^{2n+2} = A^{2n}A^2 = IA^2 = I,$$

for $A^2 = I$.

Here is a proof without induction. If the exponent is even, say $2n$, then $A^{2n} = (A^2)^n = I^n = I$, while if the exponent is odd, say $2n + 1$, then $A^{2n+1} = A^{2n}A = IA = A$ (for we just saw that $A^{2n} = I$).