Homework 4: June 30, 2011

Page67, 18 Let A and B be $n \times n$ matrices. Prove that if B is singular, then AB is singular.

By Theorem 1.5.2(ii), there is a nontrivial solution, say, $z \neq 0$, with $Bz = 0$. Multiplying by A gives $ABz = A0 = 0$; that is, the (coefficient) matrix AB of the homogeneous linear system $(AB)x = 0$ has a nontrivial solution (namely, z). Thus, Theorem 1.5.2 says that AB is singular.

Remark: We will use determinants and transpose to give a "clean" proof that BA is also singular (try proving this using only stuff in Chapter 1).

Page90, 3(h) Compute the determinant of

$$A = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ -1 & 2 & -2 & 1 \\ -3 & 2 & 3 & 1 \end{bmatrix}.$$

The easiest way is to change A into triangular form using elementary row operations [note that row echelon matrices are in (upper) triangular form]. Do some bookkeeping; keep track of the elementary operations of types I and II, for these are the ones that change the determinant. I was able to reach

$$T = \begin{bmatrix} -1 & 2 & -2 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 10 & -8 \\ 0 & 0 & & 2 \end{bmatrix}$$

using one interchange and many type III's. Fortunately, type III operations do not change the determinant, and so $\det(A) = -\det(T)$. But T is triangular, so $\det(T) = -20$ and $\det(A) = 20$.

Page97, 12 Consider the 3×3 Vandermonde matrix

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}.$$

(a) Show that $\det(V) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$.

By elementary operations of type III, we change V into

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 \\ 0 & x_3 - x_1 & x_3^2 - x_1^2 \end{bmatrix}.$$

By Laplace expansion down the first column, realizing that type III operations do not change the determinant, we have

$$\det(V) = \begin{bmatrix} x_2 - x_1 & x_2^2 - x_1^2 \\ x_3 - x_1 & x_3^2 - x_1^2 \end{bmatrix}.$$

Now we just compute the determinant of the 2×2 matrix V' to get the answer, where

$$V' = \begin{bmatrix} x_2 - x_1 & x_2^2 - x_1^2 \\ x_3 - x_1 & x_3^2 - x_1^2 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 & (x_2 - x_1)(x_2 + x_1) \\ x_3 - x_1 & (x_3 - x_1)(x_3 + x_1) \end{bmatrix}$$

Since $x^2 - y^2 = (x + y)(x - y)$, type II operations yield

$$\det(V') = (x_2 - x_1)(x_3 - x_1) \det \begin{bmatrix} 1 & x_2 + x_1 \\ 1 & x_3 + x_1 \end{bmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2),$$

for $(x_2 + x_1) - (x_3 + x_1) = x_3 - x_2$.

Remark: We sketch how to generalize this problem to larger $n \times n$ Vandermonde matrices (whose ith row is $[1, x_i, x_i^2, \ldots, x_i^{n-1}]$). First make the first column zero below the entry 1 in the (1,1) position. Then subtract row 1 from each of the other rows. Factor out $x_i - x_1$ from all the lower rows. Now do the elementary *column* operations: for all $j \geq 2$, replace column j by column $j - x_1$column 1. The $(n-1) \times (n-1)$ guy in the southeast corner is a Vandermonde matrix, and we can use induction to find its determinant.

(**b**) What conditions must the scalars x_1, x_2, x_3 satisfy in order for V to be nonsingular?

These scalars should be distinct, for then $\det(V) \neq 0$.