Homework 5: July 7, 2011

Page 98, **Ex**. **15**. Let A and B be $n \times n$ matrices. Prove that if $AB = I$, then $BA = I$. What is the significance of this result?

Since $AB = I$, we have $\det(AB) = \det(I) = 1$. Now $\det(AB) = \det(A)\det(B)$, so that $\det(A) \neq 0$ and A is nonsingular (so is B). Hence, there is a matrix C with $CA = I = AC$. But

$$C = CI = C(AB) = (CA)B = IB = B.$$

Therefore, $BA = I$.

First, this says that A and B must commute. More important, it says that to verify whether a matrix B is the inverse of A, it suffices to show only that $AB = I$, for it follows automatically that $BA = I$.

Page 116, **Ex**. **3**. Prove that the complex numbers $\mathbb{C}$ is a vector space.

There is no quick and clever proof. If nothing else, you now see the difference between a long problem and a hard problem! We must verify the two closure properties and the 8 axioms. Each of these is routine, essentially following from properties of real numbers. We will verify only two.

1. Given $z = a + ib$, there is $w = c + id$ with $z + w = 0$.

Define $c = -a$ and $d = -b$; then $(a + ib) + (c + id) = (a - a) + i(b - b) = 0$.

2. If r, s are scalars, then $(r + s)(a + ib) = r(a + ib) + s(a + ib)$.

$$\begin{aligned}(r + s)(a + ib) &= (r + s)a + i(r + s)b \\ &= (ra + sa) + i(rb + sb) \\ &= (ra + irb) + (sa + isb) \\ &= r(a + ib) + s(a + ib).\end{aligned}$$

Page 126, **Ex**. **4**. Let $x_1, \ldots, x_n$ be a spanning set for V.

(i) If we add another vector, say x_{n+1} to the set, will we still have a spanning set?

Yes. Take any $v \in V$. Since the original set spans V, there are numbers $c_1, \ldots, c_n$ with $v = c_1x_1 + \cdots + c_nx_n$. Hence, if we define $c_{n+1} = 0$, then

$$v = c_1x_1 + \cdots + c_nx_n + c_{n+1}x_{n+1} \in V.$$

(ii) If we delete one of the vectors, say x_i from the set, will we still have a spannning set?

No. We give a concrete counterexample. A spanning set for $\mathbb{R}^2$ is x_1, x_2, where $x_1 = (1, 0)$ and $x_2 = (0, 1)$. If we delete x_2, then x_1 by itself is not a spanning set. For example, there is no number c_1 with $x_2 = (0, 1) = c_1x_1 = (c_1, 0)$.