Homework 7: July 19, 2011

2.68. Let $L\colon P_4 \to P_4$ be differentiation. What is the matrix $A = {}_X[L]_X$ relative to the basis $X = 1, x, x^2, x^3, x^4$ of P_4?

The jth column of A are the "coordinates" of the derivative of the jth basis vector. For example, $L(x^3) = 3x^2$. Thus,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Page 194, **Ex**. **2**. Let $X = u_1, u_2$ and $Y = v_1, v_2$ be ordered bases for $\mathbb{R}^2$, where

$$u_1 = (1,1)^\top, \quad u_2 = (-1,1)^\top, \quad v_1 = (2,1)^\top, \quad v_2 = (1,0)^\top$$

(in class, we do not say *ordered* basis, because this is part of our definition: there is a first vector, a second vector, etc.). Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$L\colon (a,b)^\top \mapsto (-a,b)^\top.$$

Finally, the matrix $B = {}_X[L]_X = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ (see page 476).

(a). Find the transition matrix S corresponding to the change of basis from X to Y.

We know that

$$S = {}_Y[1]_X = \big({}_Y[1]_E\big)\big({}_E[1]_X\big),$$

where E is the standard basis. Now ${}_Y[1]_E$ is the inverse of the usual transition matrix V from Y to E, namely,

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix};$$

The matrix $U = {}_E[1]_X$ is the transition matrix $\left[\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right]$, and so

$$S = {}_E[1]_X = V^{-1}U = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}.$$

(b). Find the matrix A representing L wrt Y by computing SBS^{-1}.

$$SBS^{-1} = \begin{bmatrix} 1 & 0 \\ -4 & -1 \end{bmatrix}.$$

(c). Verify that

$$\begin{aligned} L(v_1) &= a_{11}v_1 + a_{21}v_2 \\ L(v_2) &= a_{12}v_1 + a_{22}v_2. \end{aligned}$$

We put in numbers (and we write all vectors wrt standard basis):

$$\begin{aligned} L(v_1) &= L(2,1) = (-2,1) = 1 \cdot (2,1) - 4(1,0) = (-2,1) \\ L(v_2) &= L(1,0) = (-1,0) = 0 \cdot (2,1) - 1 \cdot (1,0) = (-1,0). \end{aligned}$$

Page 195, **Ex**. **11**. Show that if A and B are similar matrices, then

$$\det(A) = \det(B).$$

We know that if M and N are square matrices of the same size, then $\det(MN) = \det(M)\det(N)$. Since A and B are similar, there is a nonsingular matrix S with $B = SAS^{-1}$. But we know that $\det(S^{-1}) = \det(S)^{-1}$. Hence

$$\det(B) = \det(SAS^{-1}) = \det(S)\det(A)\det(S)^{-1} = \det(A)$$

(matrices may not commute under multiplication, but numbers do).