Homework 9: July 25, 2011

Page 212, **Ex.3**. For each of the following pairs of vectors x and y, find the vector projection p of x onto y, and verify that p and $x - p$ are orthogonal.

(**a**) $x = (3,4)^\top$, $y = (1,0)^\top$

Recall that the vector projection of a vector x on a vector y is

$$p = \frac{x^T y}{y^T y} y.$$

Substitute to get $p = 3y = (3,0)^T$. Now $x - p = (0,4)^T$, and so $(p, x-p) = 3 \cdot 0 + 0 \cdot 4 = 0$.

(**d**). $x = (2,-5,4)^\top$, $y = (1,2,-1)^\top$

Substitute to get $p = -2y = (1,2,-1)^T = (-2,-4,2)^T$. Now $x - p = (4,-1,2)^T$, and so $(p, x-p) = (-2) \cdot 4 + (-4) \cdot (-1_+ 2 \cdot 2 = 0$.

Page 212, **Ex. 3**. Let S be the subspace of $\mathbb{R}^3$ spanned by the vectors $x = (x_1, x_2, x_3)^\top, y = (y_1, y_2, y_3)^\top$. Let

$$A = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}.$$

Show that $S^\perp = N(A)$.

The rows of A are x and y, and so S is the row space of A:

$$S = \{\alpha x + \beta y : \alpha, \beta \in \mathbb{R}\}.$$

Therefore, $S^\perp = \{z \in \mathbb{R}^3 : (\alpha x + \beta y, z) = 0\}$. Of course, $N(A) = \{z \in \mathbb{R}^3 : Az = 0\}$.

We show equality by proving that each of the sets is a subset of the other.

$S^\perp \subseteq N(A)$. If $z \in S^\perp$, then $(\alpha x + \beta y, z) = 0$ for all α, β; in particular, $(x,z) = 0$ and $(y,z) = 0$. But the definition of matrix multiplication says that $Az = \begin{bmatrix} (x,z) \\ (y,z) \end{bmatrix} = 0$. Hence, $z \in N(A)$.

For the reverse inclusion, suppose that $z \in N(A)$. Now $Az = 0$. As above, the definition of matrix multiplication gives $(x,z) = 0 = (y,z)$. Hence, for all scalars α, β, we have

$$(\alpha x + \beta y, z) = \alpha(x,z) + \beta(y,z) = 0,$$

and so $z \in S^\perp$.

Page 212, **Ex**. **8**. Let S be the subspace of $\mathbb{R}^n$ spanned by the vectors $x_1, \ldots, x_k$. Show that $y \in S^\perp$ if and only if $y \perp x_i$ for all i.

If $y \in S^\perp$, then $(y, s) = 0$ for every $s \in S$. In particular, $(y, x_i) = 0$ for all i.

Conversely, suppose that $(y, x_i) = 0$ for all i. If $s \in S$, then s is a linear combination of the x_i: there are scalars $c_1, \ldots, c_k$ with $s = c_1 x_1 + \cdots + c_k x_k$. Hence,

$$\begin{aligned}(y, s) &= (y, c_1 x_1 + \cdots + c_k x_k)\\ &= (y, c_1 x_1) + \cdots + (y, c_k x_k)\\ &= c_1 (y, x_1) + \cdots + c_k (y, x_k)\\ &= 0.\end{aligned}$$

Therefore, $y \in S^\perp$.