RESEARCH STATEMENT

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1. Introduction

My area of research is finite geometry. The objects of finite geometry that I study are finite affine planes (i.e., $S(2, n, n^2)$ Steiner systems, where $n > 1$). Although I could regard finite affine planes as Steiner systems and study them as combinatorial objects, I regard them as geometric objects and study them from a geometric point of view, using the Euclidean Plane as a basis for intuition and inspiration. The famous Prime Power Conjecture (PPC) motivates my research; I rely upon the objects, methods, and results of loop theory to contribute to the resolution of this conjecture.

2. Background

We first consider finite affine planes; the relevant material can be found in VI.7 of [1].

A *geometry* $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a set of *points* $\mathcal{P}$, a set of *lines* $\mathcal{L}$, and a relation $\mathcal{I}$, known as *incidence*, between them (i.e., $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$); the geometry is *finite* if both $\mathcal{P}$ and $\mathcal{L}$ are finite.

A geometry with which we are all familiar is the Euclidean Plane; although often thought of as $\mathbb{R}^2$, the Euclidean Plane may also be thought of as $\mathbb{C}$. The points of the Euclidean Plane are the elements of $\mathbb{C}$; the lines, subsets of $\mathbb{C}$ of the form $\alpha\mathbb{R} + \beta$, where $\alpha \in \mathbb{C}^\times$ and $\beta \in \mathbb{C}$; the incidence relation, set inclusion.

The Euclidean Plane has the following properties:

(1) any two distinct points are incident with exactly one line;
(2) for any point p not incident with a line l, there is exactly one line incident with p that has no point in common with l (this is the *euclidean parallel axiom*);
(3) there exist three points such that no line is incident with all three of them.

Any geometry which satisfies the three conditions above is an *affine plane*. The geometries that I study are *finite affine planes*.

For a finite affine plane $\mathbb{A}$, there is a positive integer n such that any line of $\mathbb{A}$ is incident with exactly n points. This number n is the *order* of $\mathbb{A}$. A finite affine plane of order n has n^2 points, $n^2 + n$ lines, and $n + 1$ lines incident with each point.

For a finite affine plane of order n, if we define a *parallelism* among the lines by saying that two lines are parallel if they are equal or have no point in common, then the parallelism of a finite affine plane is an equivalence relation such that any two nonparallel lines meet. The $n + 1$ equivalence classes belonging to the parallelism are *parallel classes*. Any parallel class of a finite affine plane is a partition of the point set into n lines of n points each.

For any prime power $q = p^e$, there is a *standard finite affine plane* of order q; the expression $AG(2, q)$ denotes this plane. Let $\mathbb{F}_q$ be the field of order q; let $\mathbb{F}_{q^2}$ be the quadratic extension of $\mathbb{F}_q$. The points of $AG(2, q)$ are the elements of $\mathbb{F}_{q^2}$; the lines, subsets of $\mathbb{F}_{q^2}$ of the form $\alpha\mathbb{F}_q + \beta$, where $\alpha \in \mathbb{F}_{q^2}^{\times}$ and $\beta \in \mathbb{F}_{q^2}$; the incidence relation, set inclusion.

Every known finite affine plane has prime power order. The PPC states that every finite affine plane has prime power order.

We now turn to loops; the relevant material can be found in Chapter I of [2], §4 of [3], and §2 of [4].

Suppose that $(G, \circ)$ is a set G with a binary operation $\circ$. It is easy to verify that $(G, \circ)$ is a group if and only if:

(1) for any $(a, b) \in G^2$, there exists a unique $(x, y) \in G^2$ such that $a \circ x = y \circ a = b$;
(2) if $a, b, c \in G$, then $a \circ (b \circ c) = (a \circ b) \circ c$;
(3) there exists an $e \in G$ such that, if $g \in G$, then $e \circ g = g \circ e = g$.

A *loop* $(L, \circ)$ is a nonempty set L with a binary operation $\circ$ such that $(L, \circ)$ satisfies conditions (1) and (3) above.

Suppose that $\mathbb{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is either the Euclidean Plane or one of the aforementioned standard finite affine planes. The symbol 0 denotes the zero element; the symbol $+$, normal addition; the symbol $\mathcal{F}$, the set of all lines incident with 0. We now have an ordered triple $(\mathcal{P}, \mathcal{F}, +)$ that possesses the following properties:

(1) $(\mathcal{P}, +)$ is a group whose identity element is 0;
(2) if $l \in \mathcal{F}$, then $l \subseteq \mathcal{P}$;
(3) if $l \in \mathcal{F}$, then $(l, +) \leq (\mathcal{P}, +)$ and $|l|, |\mathcal{F}| > 1$;
(4) if $p \in \mathcal{P} - \{0\}$, then there exists a unique $l \in \mathcal{F}$ such that $p \in l$;
(5) if $l \in \mathcal{F}$, then the cosets of $(l, +)$ are precisely the lines that are either equal to or disjoint from l, so the cosets of $(l, +)$ partition $\mathcal{P}$.

We regard the group $(\mathcal{P}, +)$ as part of a *fibered group* $(\mathcal{P}, \mathcal{F}, +)$ with *fibration* $\mathcal{F}$. It is possible to define analogously a *fibered loop* $(L, \mathcal{F}, \circ)$ with *fibration* $\mathcal{F}$, as long as subloops and cosets of subloops are defined for the loop $(L, \circ)$ as they would be for any group.

3. Results From Dissertation Work

Typically, mathematicians who publish results that support the PPC only focus upon special cases; these results state that finite affine planes which satisfy certain extra conditions must be of prime power order. Most of my results are similarly structured; I expect the majority of my future results to be similarly structured as well.

Suppose that $\mathbb{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a finite affine plane; we adopt the following notation:

(1) If $l_1, l_2 \in \mathcal{L}$, then $l_1 \| l_2$ indicates that l_1 and l_2 occupy the same parallel class;
(2) If $l_1, l_2 \in \mathcal{L}$, then $l_1 \nparallel l_2$ indicates that l_1 and l_2 do not occupy the same parallel class;
(3) If p and q are distinct elements of $\mathcal{P}$, then $\overline{p, q}$ denotes the unique line in $\mathcal{L}$ that is incident with both p and q;
(4) If $l_1, l_2 \in \mathcal{L}$ are such that $l_1 \nparallel l_2$, then $l_1 \cap l_2$ denotes the unique point in $\mathcal{P}$ that is incident with both l_1 and l_2;
(5) If $p \in \mathcal{P}$ and if $l \in \mathcal{L}$, then $\overline{p \| l}$ denotes the unique line in $\mathcal{L}$ that is both incident with p and in the same parallel class in $\mathbb{A}$ as l.

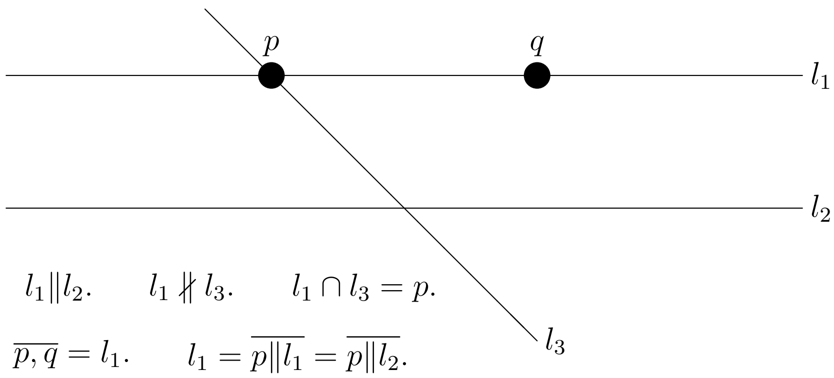

Fig.1 : An illustration which accompanies the notational definitions above.

If $o, a, b \in \mathcal{P}$ are not collinear, then $\langle a|o|b \rangle := \overline{a \| \overline{o,b}} \cap \overline{b \| \overline{o,a}} \in \mathcal{P}$.

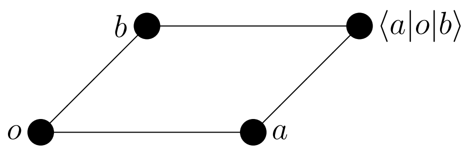

Fig.2 : An illustration which accompanies the definition above.

If $w, x, z \in \mathcal{P}$ are not collinear, then $R_w(x, z) := \langle w|z|\langle w|x|z \rangle \rangle \in \mathcal{P}$.

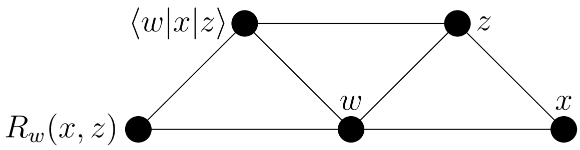

Fig.3 : An illustration which accompanies the definition above.

Suppose that a finite fibered loop $(L, \mathcal{F}, \circ)$ of order $n^2 > 1$ is such that for any $F \in \mathcal{F}$, we have that $|F| = n$ and that the left (right) cosets of F partition L; then we can use this fibered loop to create a finite affine plane of order n. The points are the elements of L; the lines, the left (right) cosets of the subloops in $\mathcal{F}$; the incidence relation, set inclusion.

Suppose that $\mathbb{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a finite affine plane of order $n > 1$. Let each line be a subset of $\mathcal{P}$, and let the incidence relation be set inclusion. Let $0_{\mathcal{P}} \in \mathcal{P}$ be a distinguished point and let $\mathcal{F} := \{l_1, ..., l_{n+1}\}$ be the set of all lines that are incident with $0_{\mathcal{P}}$. If $1 \leq i \leq n+1$, then let $\circ_i$ be a binary operation on l_i such that $(l_i, \circ_i)$ is a loop whose identity element is $0_{\mathcal{P}}$. Let $\diamond$ be a binary operation on $\mathcal{P}$ such that:

(1) if $x, y \in l_i$, then $x \diamond y = x \circ_i y$;
(2) if $0_{\mathcal{P}}, x, y$ are not collinear, then $x \diamond y = \langle x|0_{\mathcal{P}}|y \rangle$.

We now have a finite fibered loop $(\mathcal{P}, \mathcal{F}, \diamond)$ of order n^2; every subloop in $\mathcal{F}$ is both of order n and a subloop whose cosets partition $\mathcal{P}$.

The previous two paragraphs indicate that there is a strong relationship between finite affine planes and certain fibered loops of finite square order.

We refer to finite affine planes that satisfy certain extra conditions as *translation planes*; a finite affine translation plane necessarily has prime power order. Every standard finite affine plane in Section 2 is a translation plane. The geometric condition that I define below holds true in some finite affine planes and fails to hold true in other finite affine planes.

Hexagonal Condition. Suppose that $w, x, z_1, z_2 \in \mathcal{P}$ are such that $w \neq x$ and that $z_1, z_2 \notin \overline{w, x}$. Then $R_w(x, z_1) = R_w(x, z_2) \neq x$.

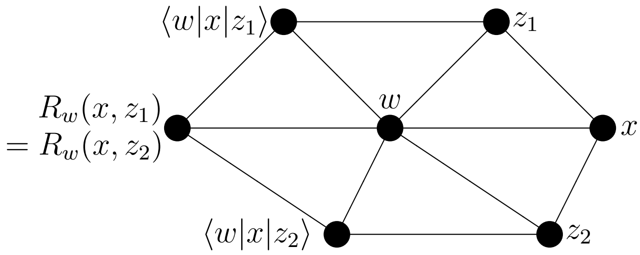

Fig.4 : An illustration of the Hexagonal Condition.

I prove in my dissertation that, for a finite affine plane $\mathbb{A}$, the Hexagonal Condition holds true for $\mathbb{A}$ if and only if $\mathbb{A}$ is a translation plane of odd order. Consequently, I use the Hexagonal Condition to characterize the finite affine translation planes of odd order. As a corollary, we have that any finite affine plane for which the Hexagonal Condition holds true is of odd prime power order.

4. Future Research

Within finite affine planes, the geometric structure allows certain loops to arise naturally. Even if these loops are not fibered loops, we can still use loop theory to study a geometric object. There is much potential for novel uses of loop theory to yield information that will contribute to the resolution of the PPC; in my research, I expect to search for those uses which yield such information.

References

[1] Colbourn, C. J. and Dinitz, J. H. (Eds.). *The CRC Handbook of Combinatorial Designs*. Boca Raton, FL: CRC Press, 1996.

[2] Pflugfelder, H. O., *Quasigroups and Loops: Introduction*, Sigma Series in Pure Mathematics **7**, Heldermann Verlag, Berlin, 1990.

[3] Zizioli, E., *Fibered Incidence Loops and Kinematic Loops*, J. Geom., **30 (2)** (1987), 144-156.

[4] Zizioli, E., *An Independence Theorem on the Conditions for Incidence Loops*, 497-502, Ann. Discrete Math. **37**, North-Holland, Amsterdam, 1988.