Math 461 Test 1, Fall 2005

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (20 points) (a) Suppose that A and B are two events with $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$. Find $P(A^c \cup B)$.
(b) Suppose that E, F, G are independent event with $P(E) = \frac{1}{2}$, $P(F) = \frac{1}{3}$ and $P(G) = \frac{1}{4}$. Find $P((E \cup F) \cap G^c)$.

(a) $P(A^c \cup B) = 1 - P(A \cap B^c) = 1 - [P(A) - P(A \cap B)] = 1 - [\frac{1}{2} - \frac{1}{4}] = \frac{3}{4}$.

(b) $P((E \cup F) \cap G^c) = P(E \cup F)P(G^c) = [P(E) + P(F) - P(E \cap F)]P(G^c) = [\frac{1}{2} + \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{3}]\frac{3}{4} = \frac{1}{2}$.

2. (15 points) 10 people, including A, B and C, are randomly arranged in a line. Find the probability that A, B and C are together in the line.

The probability that A, B and C are together in the line is

$$\frac{8!3!}{10!} = \frac{1}{15}.$$

3. (15 points) An 9-card hand is drawn without replacement from an ordinary deck of 52 cards. Find the probability that it contains all 4 cards of at least 1 of the 13 denominations.

For $i = 1, 2, \ldots, 13$, let A_i be the event that the hand contains all 4 cards of denomination i. Then we are looking for the probability of $\cup_{i=1}^{13} A_i$. Using the inclusion-exclusion formula, we get

$$\begin{aligned} P(\cup_{i=1}^{13} A_i) &= \sum_{i=1}^{13} P(A_i) - \sum_{i<j} P(A_i \cap A_j) \\ &= 13\frac{\binom{48}{5}}{\binom{52}{9}} - \binom{13}{2}\frac{\binom{44}{1}}{\binom{52}{9}}. \end{aligned}$$

4. (20 points) Box A has 5 white and 7 black balls. Box B has 3 white and 9 black balls. We flip a fair coin. If the coin comes up heads, we randomly select a ball from Box A, whereas if the coin comes up tails, we randomly select a ball from Box B. (a) Find the probability the selected ball is white. (b) Suppose that the selected ball is white, what is the probability that the coin came up tails?

Let H be teh event that the coin comes heads, T be the event that the coin comes up tails and W be the event that selected ball is white.
(a) $P(W) = P(W|H)\frac{1}{2} + P(W|T)\frac{1}{2} = \frac{5}{12}\frac{1}{2} + \frac{3}{12}\frac{1}{2} = \frac{1}{3}$.
(b) $P(T|W) = \frac{P(WT)}{P(W)} = \frac{P(W|T)P(T)}{P(W)} = \frac{3/24}{8/24} = \frac{3}{8}$.

5. (15 points) A box contains 4 red balls and 6 black balls. Players A and B draw balls from the box consecutively until a red ball is selected. Find the probability that B gets the red ball. (A draws first ball, then B and so on. There is no replacement of the balls drawn.)

The only cases for B to get the red ball are BR, $BBBR$ and $BBBBBR$. Therefore the answer is

$$\frac{6}{10}\frac{4}{9}+\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{4}{7}+\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{4}{5}.$$

6. (15 points) We draw cards, one at a time, at random and successively from an ordinary deck of 52 cards with replacement. Find the probability that an ace appears before a face card (that is, a Jack, a Queen or a King)?

Let A be the event the first drawn card is an ace, B the event that the first drawn card is a face card, and C teh event that the first drawn card is neither an ace nor a face card. Let E be the event that an ace appears before a face card. Then, conditioning on the first draw, we get

$$P(E)=P(E|A)\frac{1}{13}+P(E|B)\frac{3}{13}+P(E|C)\frac{9}{13}=\frac{1}{13}+P(E)\frac{9}{13}.$$

Thus $P(E)=\frac{1}{4}$.