Math 461 Test 2, Fall 2005

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (9 points) The distribution function of a random variable X is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ x/4 & 0 \le x < 1 \\ 1/2 & 1 \le x < 2 \\ \frac{1}{12}x + \frac{1}{2} & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

Find (a) $P(1 \le X < 3)$, (b) $P(X > \frac{3}{2})$, (c) $P(2 < X \le 7)$.

(a) $P(1 \le X < 3) = F(3-) - F(1-) = (\frac{1}{4} + \frac{1}{2}) - \frac{1}{4} = \frac{1}{2}$.
(b) $P(X > \frac{3}{2}) = 1 - F(\frac{3}{2}) = \frac{1}{2}$.
(c) $(2 < X \le 7) = F(7) - F(2) = 1 - (\frac{1}{6} + \frac{1}{2}) = \frac{1}{3}$.

2. (11 points) Suppose that X is a uniform random variable on $(-1, 1)$. Find the density of the random variable $Y = X^4$.

Y takes values in $(0, 1)$. For any $y \in (0, 1)$,

$$P(Y \le y) = P(X^4 \le y) = P(-y^{1/4} \le X \le y^{1/4}) = y^{1/4}.$$

Thus the density of Y is given by

$$g(y) = \begin{cases} \frac{1}{4}y^{-3/4} & y \in (0, 1) \\ 0 & \text{otherwise} \end{cases}$$

3. (20 points) A certain flight has 90 seats for passengers. Assume that any passenger has an 80% probability of showing up for that flight (with different passengers mutually independent). Suppose that 100 tickets are sold. Use the normal approximation to find the probability that the flight can accommodate all passengers (with a ticket) that show up.

Let X be the number of ticketed passengers who show up for the flight, then X is a binomial random variable with parameters $(100, .8)$. Thus by normal approximation we get that

$$P(X \le 90) = P(X < 90.5) = P(\frac{X - 80}{4} \le \frac{90.5 - 80}{4}) = \Phi(2.62) = .9955.$$

4. (20 points) Two teams play a series of games and they stop playing as soon as one of the teams wins 4 games. Suppose that the two teams are evenly matched and each has probability $\frac{1}{2}$ of winning each game. Find the expected number of games played.

$$P(X=4) = 2 \cdot (\frac{1}{2})^4 = \frac{1}{8}$$
$$P(X=5) = 2 \cdot \begin{pmatrix} 4 \\ 1 \end{pmatrix} (\frac{1}{2})^5 = \frac{1}{4}$$
$$P(X=6) = 2 \cdot \begin{pmatrix} 5 \\ 2 \end{pmatrix} (\frac{1}{2})^6 = \frac{5}{16}$$
$$P(X=7) = 2 \cdot \begin{pmatrix} 6 \\ 3 \end{pmatrix} (\frac{1}{2})^7 = \frac{5}{16}$$

Thus
$$E[X] = 4\frac{1}{8} + 5\frac{1}{4} + 6\frac{5}{16} + 7\frac{5}{16} = \frac{93}{16}.$$

5. (20 points) Let X be an exponential random variable with parameter $\lambda = \ln 3$. Define a positive integer valued random variable Y letting $Y = n$ whenever $n-1 < X \le n$ for any positive integer n. (a) Find the mass function of Y; (b) find $E[Y]$ and $\mathrm{Var}(Y)$.

(a) For any positive integer n,

$$P(Y=n) = P(n-1 < X \le n) = e^{-(n-1)\ln 3} - e^{-n \ln 3} = (\frac{1}{3})^{n-1} - (\frac{1}{3})^n = (\frac{1}{3})^{n-1}\frac{2}{3}.$$

(b) From part (a) we know that Y is a geometric random variable with parameter $p = \frac{2}{3}$, thus $E[y] = \frac{3}{2}$ and $\mathrm{Var}(Y) = \frac{3}{4}$.

6. (20 points) The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} ye^{-2x} & x > 0, 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal density of X; (b) find $E[X]$ and $\mathrm{Var}(X)$.

(a) The marginal density of X is

$$f_X(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) From part (a) we know that X is an exponential random variable with parameter $\lambda = 2$. Thus $E[X] = \frac{1}{2}$ and $\mathrm{Var}(X) = \frac{1}{4}$.