

Twelfth Homework Set — Solutions

Chapter 7

Problem 5 If (X, Y) is the location of the accident, then X and Y are uniform random variables on $(-\frac{3}{2}, \frac{3}{2})$. Let $D = |X| + |Y|$. Then

$$\begin{aligned} E[D] &= E[|X|] + E[|Y|] = 2E[|X|] \\ &= 2 \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{|x|}{3} dx = \frac{4}{3} \int_0^{\frac{3}{2}} x dx \\ &= \frac{4}{3} \cdot \frac{9}{8} = \frac{3}{2}. \end{aligned}$$

Problem 6 Let X_i be the outcome of the i -th roll of the die, for $i = 1, \dots, 10$, and note that $E[X_i] = \frac{7}{2}$. Let $X = X_1 + \dots + X_{10}$. Now $E[X] = E[X_1] + \dots + E[X_{10}] = 10E[X_1] = 35$.

Problem 7 (a) Let X_i be one if both A and B choose the i -th object, for $i = 1, \dots, 10$. Then $E[X_i] = P\{X_i = 1\} = \left(\frac{3}{10}\right)^2 = \frac{9}{100}$. Now, the expected number of objects chosen by both A and B is $E[X_1] + \dots + E[X_{10}] = 10E[X_1] = 0.9$.

(b) Let Y_i be one if neither A nor B choose the i -th object. Then $E[Y_i] = P\{Y_i = 1\} = \left(\frac{7}{10}\right)^2 = \frac{49}{100}$, so that $E[Y_1 + \dots + Y_{10}] = 10E[Y_1] = 4.9$.

(c) Let Z_i be one if either A or B (but not both) chooses the i -th object. Then $E[Z_i] = P\{Z_i = 1\} = 2\frac{3}{10}\frac{7}{10} = \frac{21}{50}$. Now, $E[Z_1 + \dots + Z_{10}] = 10E[Z_1] = \frac{21}{5} = 4.2$.

Problem 8 Following the hint, let X_i be one if the i -th arrival sits at a previously unoccupied table. Then $E[X_i] = P\{X_i = 1\} = (1-p)^{i-1}$, so that

$$E[X_1 + \dots + X_N] = \sum_{i=1}^N (1-p)^{i-1} = \frac{1 - (1-p)^N}{1 - (1-p)} = \frac{1 - (1-p)^N}{p}.$$

Problem 11 Let X_i be one if the i -th outcome differs from the $(i-1)$ -th outcome, for $i = 2, \dots, n$. We have $E[X_i] = P\{X_i = 1\} = 2p(1-p)$, so that $E[X_2 + \dots + X_n] = 2(n-1)p(1-p)$.

Problem 18 Let X_i be one if the i -th card is a match, for $i = 1, \dots, 13$, and let $X = X_1 + \dots + X_{52}$. Then $P\{X_i = 1\} = \frac{1}{13}$, so that $E[X] = 52E[X_1] = \frac{52}{13} = 4$.

Problem 19 (a) If X is the number of insects caught before a type 1 catch, then $(X + 1)$ is geometric with parameter P_1 , so that $E[X] = \frac{1}{P_1} - 1$.

(b) Let Y_i be one if an insect of type i is caught before an insect of type 1, for $i = 2, \dots, r$. Then $Y = Y_2 + \dots + Y_r$ is the number of insects caught before an insect of type 1. We have $E[Y_i] = P\{Y_i = 1\} = \frac{P_i}{P_i + P_1}$, so that

$$E[Y] = \sum_{i=2}^r \frac{P_i}{P_i + P_1}.$$

Problem 21 (a) Let X be the number of days of the year that are birthdays of exactly 3 people. For $i = 1, \dots, 365$, let $X_i = 1$ if the i -day is the birthday of exactly 3 people and $X_i = 0$ otherwise. Then $X = \sum_{i=1}^{365} X_i$. Since for each i ,

$$EX_i = P(X_i = 1) = \binom{100}{3} \left(\frac{1}{365}\right)^3 \left(\frac{364}{365}\right)^{97},$$

we get that

$$EX = 365 \binom{100}{3} \left(\frac{1}{365}\right)^3 \left(\frac{364}{365}\right)^{97}.$$

(b) Let Y be the number of distinct birthdays. For $i = 1, \dots, 365$, let $Y_i = 1$ if the i -day is someone's birthday and $Y_i = 0$ otherwise. Then $Y = \sum_{i=1}^{365} Y_i$. Since for each i ,

$$EY_i = P(Y_i = 1) = 1 - P(Y_i = 0) = 1 - \left(\frac{364}{365}\right)^{100},$$

we get that

$$EY = 365 \left[1 - \left(\frac{364}{365}\right)^{100} \right].$$

Problem 30 Note that $E[X^2] = E[Y^2] = \text{Var}(X) + E[X]^2 = \sigma^2 + \mu^2$. Now we conclude that

$$E[(X - Y)^2] = E[X^2] - 2E[X]E[Y] + E[Y^2] = 2\sigma^2,$$

using the fact that X and Y are independent.

Problem 31 Let X_i be the outcome of the i -th roll of the die, for $i = 1, \dots, 10$. Then $\text{Var}(X_i) = \frac{35}{12}$, so that

$$\text{Var}(X_1 + \dots + X_{10}) = 10 \cdot \frac{35}{12} = \frac{175}{6}.$$

Problem 33 (a)

$$E[(2 + X)^2] = 4 + 4E[X] + E[X^2] = 8 + \text{Var}(X) + E[X]^2 = 14.$$

(b)

$$\text{Var}(4 + 3X) = 9\text{Var}(X) = 45.$$

Problem 38 We have

$$\begin{aligned} E[XY] &= \int_0^\infty \int_0^x 2ye^{-2x} dy dx = \int_0^\infty x^2 e^{-2x} dx = \frac{1}{4}, \\ E[X] &= \int_0^\infty \int_0^x 2e^{-2x} dy dx = \int_0^\infty 2xe^{-2x} dx = \frac{1}{2}, \quad \text{and} \\ E[Y] &= \int_0^\infty \int_0^x \frac{2y}{x} e^{-2x} dy dx = \int_0^\infty xe^{-2x} dx = \frac{1}{4}. \end{aligned}$$

Hence,

$$\text{Cov}(X, Y) = \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$