Twelfth Homework Set — Solutions

Chapter 7

Problem 39 We have

$$\begin{aligned}\text{Cov}(Y_n, Y_n) &= \text{Var}(Y_n) = 3\sigma^2,\\ \text{Cov}(Y_n, Y_{n+1}) &= \text{Cov}(X_n + X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2} + X_{n+3})\\ &= \text{Cov}(X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2})\\ &= \text{Var}(X_{n+1} + X_{n+2}) = 2\sigma^2,\\ \text{Cov}(Y_n, Y_{n+2}) &= \text{Cov}(X_n + X_{n+1} + X_{n+2}, X_{n+2} + X_{n+3} + X_{n+4})\\ &= \text{Cov}(X_{n+2}, X_{n+2}) = \text{Var}(X_{n+2}) = \sigma^2, \quad \text{and}\\ \text{Cov}(Y_n, Y_{n+j}) &= 0 \quad \text{if } j \geq 3.\end{aligned}$$

Problem 41 the number of carp is a hypergeometric random variable, so that we have

$$E[X] = \frac{20 \cdot 30}{100} = 6,$$

and

$$\text{Var}(X) = \frac{20 \cdot 80}{99} \cdot \frac{3}{10} \cdot \frac{7}{10} = \frac{112}{33}.$$

Problem 42 (a) Let X_i be one if the i-th pair consists of a man and a women, and zero otherwise. Then the sum $X_1 + \cdots + X_{10}$ is the number of pairs that consist of a man and a woman.

We have $E[X_i] = P\{X_i = 1\} = 2 \cdot \frac{10 \cdot 10}{20 \cdot 19} = \frac{10}{19}$, so that

$$E[X_1 + \cdots + X_{10}] = \frac{100}{19}.$$

Now, we have $\text{Var}(X_i) = E[X_i^2] - E[X_i]^2 = \frac{10}{19} - \frac{100}{361} = \frac{90}{361}$, and $\text{Cov}(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j] = \frac{10}{19} \cdot \frac{9}{17} - \frac{100}{361} = \frac{10}{6137}$ if $i \neq j$, so that

$$\text{Var}(X_1 + \cdots + X_{10}) = \frac{900}{361} + 10 \cdot 9 \cdot \frac{10}{6137} = \frac{16200}{6137} = 2.6397.$$

(b) Let Y_i be one if the i-th couple are paired together. $E[Y_i] = P\{Y_i = 1\} = \frac{2 \cdot 10 \cdot 18!}{20!} = \frac{1}{19}$, so that

$$E[Y_1 + \cdots + Y_{10}] = \frac{10}{19}.$$

We have $\text{Var}(Y_i) = E[Y_i^2] - E[Y_i]^2 = \frac{1}{19} - \frac{1}{361} = \frac{18}{361}$ and $E[Y_iY_j] = \frac{8\binom{10}{2}\cdot 16!}{20!} = \frac{1}{323}$, so that $\text{Cov}(Y_i, Y_j) = \frac{1}{323} - \frac{1}{361} = \frac{2}{6137}$, so that

$$\text{Var}(Y_1 + \cdots + Y_{10}) = \frac{180}{361} + 90 \cdot \frac{2}{6137} = \frac{3240}{6137}.$$

Problem 50 We have

$$f_Y y = \int_0^\infty \frac{e^{-\frac{x}{y}} - y}{y} dx = e^{-y}$$

for $y > 0$, so that

$$f_{X|Y}(x|y) = \begin{cases} \frac{e^{-\frac{x}{y}}}{y} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Now, we have

$$E\left[X^2|Y\right] = \int_0^\infty \frac{x^2}{y} e^{-\frac{x}{y}} dx = 2y^2.$$

Problem 51 We have

$$f_Y(y) = \int_0^y \frac{e^{-y}}{y} dx = e^{-y},$$

so that

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} & x \in (0, y) \\ 0 & \text{otherwise.} \end{cases}$$

We conclude that

$$E\left[X^3|Y = y\right] = \int_0^y \frac{x^3}{y} dx = \frac{y^3}{4}.$$

Problem 56 Let Y_i be one if the elevator stops at the i-th floor, for $i = 1, \ldots, N$. Let $Y = Y_1 + \cdots + Y_{10}$. Let X be the number of passengers, i.e., X is Poisson with parameter 10. We have $E[Y_i = 1|X = k] = 1 - \left(\frac{N-1}{N}\right)^k$, so that

$$E[Y|X = k] = N\left(1 - \left(\frac{N-1}{N}\right)^k\right).$$

We have

$$
\begin{aligned}
E[Y] = E[E[Y|X]] &= E\left[N\left(1-\left(\frac{N-1}{N}\right)^k\right)\right] \\
&= N - N\sum_{k=0}^{\infty}\left(\frac{N-1}{N}\right)^k \frac{10^k}{k!}e^{-10} \\
&= N(1-e^{-\frac{10}{N}}).
\end{aligned}
$$

Problem 57 By Example 4d in Section 7.4, we have

$$
E\left[\sum_{i=1}^{N} X_i\right] = E[N]\,E[X_1] = 12.5.
$$

Problem 75 X is a random variable with moment generating function $M_X(t) = \exp\{2e^t - 2\} = \exp\{2(e^t-1)\}$, i.e., X is Poisson with parameter $\lambda = 2$.

Y is a random variable with moment generating function $M_Y(t) = (\frac{3}{4}e^t + \frac{1}{4})^{10}$, i.e., Y is binomial with parameters $(10, \frac{3}{4})$.

(a)

$$
\begin{aligned}
P\{X+Y=2\} &= P\{X=0\}\,P\{Y=2\} + P\{X=1\}\,P\{Y=1\} \\
&\quad + P\{X=2\}\,P\{Y=0\} \\
&= e^{-2}\cdot\binom{10}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^8 + 2e^{-2}\cdot 10\frac{3}{4}\left(\frac{1}{4}\right)^9 \\
&\quad + 2e^{-2}\cdot\left(\frac{1}{4}\right)^{10} \\
&= e^{-2}\left(\frac{1}{4}\right)^{10}(405+60+2) = \frac{467}{4^{10}e^2}.
\end{aligned}
$$

(b)

$$
\begin{aligned}
P\{XY=0\} &= P\{X=0\} + P\{Y=0\} - P\{X=0\}\,P\{Y=0\} \\
&= e^{-2} + \frac{1}{4^{10}} - e^{-2}\frac{1}{4^{10}} = \frac{4^{10}+e^2-1}{4^{10}e^2}.
\end{aligned}
$$

(c)

$$\begin{aligned} E[XY] &= E[X] \cdot E[Y] \quad \text{by independence} \\ &= 2 \cdot 7.5 \\ &= 15. \end{aligned}$$

Chapter 8

Problem 1 $P(0 \leq X \leq 40) = 1 - P(|X - 20| > 20) \geq 1 - \frac{20}{400} = \frac{19}{20}$.

Problem 2 (a) $P(X \geq 85) \leq \frac{75}{85} = \frac{15}{17}$.

(b) $P(65 \leq X \leq 85) = 1 - P(|X - 75| > 10) \geq 1 - \frac{25}{100} = \frac{3}{4}$.

(c) Since

$$P\left(|\sum_{i=1}^{n} \frac{X_i}{n} - 75| > 5\right) \leq \frac{25}{25n},$$

we need $n = 10$.

Problem 4 (a) $P(\sum_{i=1}^{20} X_i > 15) \leq \frac{20}{15}$.

(b)

$$\begin{aligned} P(\sum_{i=1}^{20} X_i > 15) &= P(\sum_{i=1}^{20} X_i > 15.5) \approx P(Z > \frac{15.5 - 20}{\sqrt{20}} \\ &= P(Z > -1.006) \approx .8428. \end{aligned}$$

Problem 5 Let X_i be the i-th roundoff erroe, then $E(\sum_{i=1}^{50} X_i) = 0$ and $\text{Var}(\sum_{i=1}^{50} X_i) = \frac{50}{12}$. Hence by the central limit theorem

$$P\left(|\sum_{i=1}^{50} X_i| > 3\right) \approx P(|Z| > \frac{3}{\sqrt{12/50}}) = 2P(Z > 1.47) = .1416.$$

Problem 8 If we let X_i be the lifetime of the i-th light bulb and R_i be the time to replace the i-th light bulb, then the desired probability is

$$P\left(\sum_{i=1}^{100} X_i + \sum_{i=1}^{99} R_i \leq 550\right).$$

It follows from the central limit theorem that $\sum_{i=1}^{100} X_i$ is approximately a normal random variable with mean 500 and variance 2500 and that $\sum_{i=1}^{99} R_i$ is approximately a normal random variable with mean 24.75 and variance 99/48, therefore $\sum_{i=1}^{100} X_i + \sum_{i=1}^{99} R_i$ is approximately a normal random variable with mean 524.75 and variance 2502.02. Consequently the desired probability is equal to

$$P(Z \leq \frac{550 - 524.75}{\sqrt{2502.02}} = P(Z \leq .505) = .693.$$