

Third Homework Set — Solutions

Chapter 2

Problem 37 (a) There are $\binom{10}{5}$ selections for the final exam. The number of selections that allow the student to solve all problems is $\binom{7}{5}$, so that the desired probability is $\frac{\binom{7}{5}}{\binom{10}{5}} = 0.08333$.

(b) There are $\binom{7}{4} \cdot \binom{3}{1}$ selections that'll let the student solve exactly four problems, so that the probability of solving at least four problems is $\frac{\binom{7}{5} + \binom{7}{4} \cdot \binom{3}{1}}{\binom{10}{5}} = \frac{1}{2}$.

Problem 43 (a) There are $n!$ ways to arrange n people in a line. There are $2(n-1)!$ ways to arrange them such that A and B are next to each other. Hence, the probability of A and B being next to each other is $\frac{2(n-1)!}{n!} = \frac{2}{n}$.

(b) If $n = 2$, then A and B will always be next to each other. Now, assume that $n > 3$. After A picks a seat, there are $n - 1$ seats left, two of which are next to A , so that the desired probability is $\frac{2}{n-1}$.

Problem 50 The probability that you have five spades and your partner has the remaining eight spades is

$$\frac{\binom{13}{5} \binom{39}{8} \binom{8}{8} \binom{31}{5}}{\binom{52}{13,13,26}} = 2.6084 \cdot 10^{-6}.$$

Problem 53 Let E_i be the event that the i -th couple sit together, for $j = 1, \dots, 4$. Then $P(E_i) = \frac{2}{8} = \frac{1}{4}$ (Problem 43(a)). Moreover, if $i < j$, then $P(E_i E_j) = \frac{2^2 \cdot 6!}{8!}$. Similarly, if $i < j < k$, then $P(E_i E_j E_k) = \frac{2^3 \cdot 5!}{8!}$. Finally, we have $P(E_1 E_2 E_3 E_4) = \frac{2^4 \cdot 4!}{8!}$. Using inclusion-exclusion, we obtain

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3 \cup E_4) &= \sum_{i=1}^4 P(E_i) - \sum_{i < j} P(E_i E_j) + \sum_{i < j < k} P(E_i E_j E_k) - P(E_1 E_2 E_3 E_4) \\ &= 4 \cdot \frac{1}{4} - \binom{4}{2} \frac{2^2 \cdot 6!}{8!} + \binom{4}{3} \frac{2^3 \cdot 5!}{8!} - \frac{2^4 \cdot 4!}{8!} \\ &= 1 - \frac{3}{7} + \frac{2}{21} - \frac{1}{105} = \frac{23}{35}. \end{aligned}$$

Hence, the probability that no husband sits next to his wife is $1 - \frac{23}{35} = \frac{12}{35}$.

Problem 54 Let S , H , C , and D be the event that spades are missing, hearts are missing, etc. Then

$$\begin{aligned} P(S \cup H \cup C \cup D) &= P(S) + P(H) + P(C) + P(D) \\ &\quad - P(SH) - P(SC) - P(SD) - P(HC) - P(HD) - P(CD) \\ &\quad + P(SHC) + P(SHD) + P(SCD) + P(HCD) \\ &\quad - P(SHCD) \\ &= 4 \cdot \frac{\binom{39}{13}}{\binom{52}{13}} - 6 \cdot \frac{\binom{26}{13}}{\binom{52}{13}} + 4 \cdot \frac{1}{\binom{52}{13}} - 0 \\ &= 0.0511. \end{aligned}$$

Chapter 3

Problem 1 Let E be the event that at least one die lands on six, and let F be the event that the dice land of different numbers. Then $P(EF) = 2 \cdot \frac{1}{6} \cdot 56 = \frac{5}{18}$ and $P(F) = \frac{30}{36} = \frac{5}{6}$. Hence,

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\frac{5}{18}}{\frac{5}{6}} = \frac{1}{3}.$$

Problem 5

$$\frac{6 \cdot 5 \cdot 9 \cdot 8}{15 \cdot 14 \cdot 13 \cdot 12} = \frac{6}{91}$$

Problem 6 Let A be the event that the sample drawn contains exactly three white balls. Let B be the event that the first and third ball drawn are white.

Without replacement

$$P(A) = 4 \frac{8 \cdot 7 \cdot 6 \cdot 4}{12 \cdot 11 \cdot 10 \cdot 9} \text{ and } P(AB) = 2 \frac{8 \cdot 7 \cdot 6 \cdot 4}{12 \cdot 11 \cdot 10 \cdot 9}, \text{ hence } P(B|A) = \frac{P(AB)}{P(A)} = \frac{1}{2}.$$

With replacement

$$P(A) = \binom{4}{3} \left(\frac{2}{3}\right)^3 \frac{1}{3} \text{ and } P(AB) = 2 \left(\frac{2}{3}\right)^3 \frac{1}{3}, \text{ hence } P(B|A) = \frac{P(AB)}{P(A)} = \frac{1}{2}.$$

Problem 9 Let E_i be the event that the ball drawn from the i -th urn is white, for $i = 1, 2, 3$. Let F be the event that exactly two white balls were drawn. Then

$$\begin{aligned}
 P(E_1|F) &= \frac{E_1F}{P(F)} \\
 &= \frac{P(E_1E_2E_3^c) + P(E_1E_2^cE_3)}{P(E_1E_2E_3^c) + P(E_1E_2^cE_3) + P(E_1^cE_2E_3)} \\
 &= \frac{\frac{2 \cdot 8 \cdot 3}{6 \cdot 12 \cdot 4} + \frac{2 \cdot 4 \cdot 1}{6 \cdot 12 \cdot 4}}{\frac{2 \cdot 8 \cdot 3}{6 \cdot 12 \cdot 4} + \frac{2 \cdot 4 \cdot 1}{6 \cdot 12 \cdot 4} + \frac{4 \cdot 8 \cdot 1}{6 \cdot 12 \cdot 4}} \\
 &= \frac{7}{11}.
 \end{aligned}$$

Problem 10 Let E be the event that the first card is a spade, and let F be the event that the second and third cards are spades. Then $P(EF) = \frac{\binom{13}{3}}{\binom{52}{3}}$ and $P(F) = \frac{\binom{13}{3} + \binom{13}{2}\binom{39}{1}}{\binom{52}{3}}$. Hence,

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\binom{13}{3}}{\binom{13}{3} + 39\binom{13}{2}} = \frac{11}{128}.$$