

Fourth Homework Set — Solutions

Chapter 3

Problem 20 (a) $P(F|C) = \frac{P(FC)}{P(C)} = \frac{.02}{.05} = \frac{2}{5}$.

(b) $P(C|F) = \frac{P(FC)}{P(F)} = \frac{.02}{.52} = \frac{1}{26}$.

Problem 23 (a) Let W be the event that the ball selected urn II is white, E be the event that the transferred ball is white and F be the event that the transferred ball is red, then

$$P(W) = P(E)P(W|E) + P(F)P(W|F) = \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{9}.$$

(b)

$$P(E|W) = \frac{P(EW)}{P(W)} = \frac{P(E)P(W|E)}{P(W)} = \frac{1}{2}.$$

Problem 30 Let B and W be the events that the marble is black and white respectively, and let B_i be the event that box i is chosen. Then

$$P(B) = P(B_1)P(B|B_1) + P(B_2)P(B|B_2) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{2}{3} = \frac{2}{3},$$

$$P(B_1|W) = \frac{P(B_1W)}{P(W)} = \frac{P(B_1)P(W|B_1)}{P(W)} = \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{5}{12}} = \frac{3}{5}.$$

Problem 47 (a)

$$P(\text{all white}) = \frac{1}{6} \left(\frac{5}{15} + \frac{5}{15} \cdot \frac{4}{14} + \frac{5}{15} \cdot \frac{4}{14} \cdot \frac{3}{13} + \frac{5}{15} \cdot \frac{4}{14} \cdot \frac{3}{13} \cdot \frac{2}{12} + \frac{5}{15} \cdot \frac{4}{14} \cdot \frac{3}{13} \cdot \frac{2}{12} \cdot \frac{1}{11} \right).$$

(b)

$$P(3|\text{all white}) = \frac{\frac{1}{6} \cdot \frac{5}{15} \cdot \frac{4}{14} \cdot \frac{3}{13}}{P(\text{all white})}.$$

Problem 51 Let R be the event that she receives a job offer, S be the event that event of a strong recommendation, M the event of a moderate recommendation and W the event of a weak recommendation.

(a)

$$\begin{aligned} P(R) &= P(S)P(R|S) + P(M)P(R|M) + P(W)P(R|W) \\ &= (.8)(.7) + (.4)(.2) + (.1)(.1) = .65. \end{aligned}$$

(b)

$$P(S|R) = \frac{P(SR)}{P(R)} = \frac{P(S)P(R|S)}{P(R)} = \frac{(.8)(.7)}{.65} = \frac{56}{65}.$$

Similarly

$$P(M|R) = \frac{8}{65}, \quad P(W|R) = \frac{1}{65}.$$

Problem 56

$$P(\text{new}) = \sum_{i=1}^m p_i P(\text{new} | \text{the } n\text{-th coupon is of type } i) = \sum_{i=1}^m p_i (1-p_i)^{n-1}.$$

Problem 57 (a) $P(\text{original price after two days}) = \binom{2}{1} p(1-p) = 2p(1-p)$

(b) $P(\text{increase by one after three days}) = \binom{3}{2} p^2(1-p) = 3p^2(1-p)$

(c) $P(\text{increase on first day} | \text{increase by one after three days}) = \frac{p \cdot 2p(1-p)}{3p^2(1-p)} = \frac{2}{3}$

Problem 59 (a) $P(HHHH) = p^4$

(b) $P(THHH) = p^3(1-p)$

(c) The pattern $HHHH$ can *only* occur before $THHH$ if the first four coin flips come up heads. Hence, $P(THHH \text{ occurs before } HHHH) = 1 - p^4$.

Problem 64 Let E be the event that the wife answers correctly, and let F be the event that the husband answers correctly.

(a) If only one of them answers, then the probability of a correct answer is $P(E) = P(F) = p$.

(b) $P(\text{correct answer}) = P(EF) + \frac{1}{2} \cdot 2 \cdot p(1-p) = p^2 + p - p^2 = p$

Problem 66 Let E_i be the event that the i -th switch is on.

(a)

$$\begin{aligned} & P(\text{current flows from } A \text{ to } B) \\ &= (P(E_1E_2) + P(E_3E_4) - P(E_1E_2E_3E_4)) P(E_5) \\ &= (p_1p_2 + p_3p_4 - p_1p_2p_3p_4)p_5 \end{aligned}$$

(b)

$$\begin{aligned} & P(\text{current flows from } A \text{ to } B) \\ &= P(E_1E_4 \cup E_1E_3E_5 \cup E_2E_5 \cup E_2E_3E_4) \\ &= p_1p_4 + p_1p_3p_5 + p_2p_5 + p_2p_3p_4 \\ &\quad - p_1p_3p_4p_5 - p_1p_2p_4p_5 - p_1p_2p_3p_4 \\ &\quad - p_1p_2p_3p_5 - p_1p_2p_3p_4p_5 - p_2p_3p_4p_5 \\ &\quad + 4p_1p_2p_3p_4p_5 - p_1p_2p_3p_4p_5 \\ &= p_1p_4 + p_1p_3p_5 + p_2p_5 + p_2p_3p_4 \\ &\quad - p_1p_3p_4p_5 - p_1p_2p_4p_5 - p_1p_2p_3p_4 \\ &\quad - p_1p_2p_3p_5 - p_2p_3p_4p_5 + 2p_1p_2p_3p_4p_5 \end{aligned}$$

Problem 78 (a) $P(\text{exactly four games are played}) = P(ABAA) + P(BAAA) + P(ABBB) + P(BABB) = 2p^3(1-p) + 2p(1-p)^3 = 2p(1-p)(p^2 + (1-p)^2) = 2p(1-p)(1-2p+2p^2)$

(b) Let E be the event that A wins the match. Conditioning on the first two games of the match, we get $P(E) = P(E|A, A)p^2 + P(E|A, B)p(1-p) + P(E|B, A)(1-p)p + P(E|B, B)(1-p)^2 = p^2 + 2P(E)p(1-p)$ because $P(E|A, B) = P(E|B, A) = P(E)$. Hence, $P(E) = \frac{p^2}{1-2p(1-p)}$.

Problem 81 Using the gambler's ruin formula, the answer is

$$\frac{1 - (9/11)^{15}}{1 - (9/11)^{30}}$$

Problem 83 (a) Conditioning on the coin flip

$$P(\text{throw } n \text{ is red}) = \frac{1}{2} \frac{4}{6} + \frac{1}{2} \frac{2}{6} = \frac{1}{2}$$

(b)

$$P(R_3|R_1R_2) = \frac{P(R_1R_2R_3)}{P(R_1R_2)} = \frac{\frac{1}{2}\left(\frac{2}{3}\right)^3 + \frac{1}{2}\left(\frac{1}{3}\right)^3}{\frac{1}{2}\left(\frac{2}{3}\right)^2 + \frac{1}{2}\left(\frac{1}{3}\right)^2} = \frac{3}{5}.$$

(c)

$$P(A|R_1R_2) = \frac{P(R_1R_2|A)P(A)}{P(R_1R_2)} = \frac{\left(\frac{2}{3}\right)^{2\frac{1}{2}}}{\left(\frac{2}{3}\right)^{2\frac{1}{2}} + \left(\frac{1}{3}\right)^{2\frac{1}{2}}} = \frac{4}{5}.$$

Problem 84 (a)

$$\begin{aligned} P(A \text{ win}) &= \sum_{i=0}^{\infty} P(\text{ the first white appears on draw number } 3i + 1) \\ &= \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{3i} \frac{1}{3} = \frac{1}{3} \sum_{i=0}^{\infty} \left(\frac{8}{27}\right)^i = \frac{1}{3} \frac{1}{1 - \frac{8}{27}} \end{aligned}$$

$$\begin{aligned} P(B \text{ win}) &= \sum_{i=0}^{\infty} P(\text{ the first white appears on draw number } 3i + 2) \\ &= \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{3i+1} \frac{1}{3} = \frac{2}{9} \sum_{i=0}^{\infty} \left(\frac{8}{27}\right)^i = \frac{2}{9} \frac{1}{1 - \frac{8}{27}} \end{aligned}$$

$$\begin{aligned} P(C \text{ win}) &= \sum_{i=0}^{\infty} P(\text{ the first white appears on draw number } 3i + 3) \\ &= \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{3i+2} \frac{1}{3} = \frac{4}{27} \sum_{i=0}^{\infty} \left(\frac{8}{27}\right)^i = \frac{4}{27} \frac{1}{1 - \frac{8}{27}} \end{aligned}$$

(b)

$$\begin{aligned} P(A \text{ win}) &= \frac{4}{12} + \frac{8}{12} \frac{7}{11} \frac{6}{10} \frac{4}{9} + \frac{8}{12} \frac{7}{11} \frac{6}{10} \frac{5}{9} \frac{4}{8} \frac{3}{7} \frac{4}{6} \\ P(B \text{ win}) &= \frac{8}{12} \frac{4}{11} + \frac{8}{12} \frac{7}{11} \frac{6}{10} \frac{5}{9} \frac{4}{8} + \frac{8}{12} \frac{7}{11} \frac{6}{10} \frac{5}{9} \frac{4}{8} \frac{3}{7} \frac{2}{6} \frac{4}{5} \\ P(C \text{ win}) &= \frac{8}{12} \frac{7}{11} \frac{4}{10} + \frac{8}{12} \frac{7}{11} \frac{6}{10} \frac{5}{9} \frac{4}{8} \frac{4}{7} + \frac{8}{12} \frac{7}{11} \frac{6}{10} \frac{5}{9} \frac{4}{8} \frac{3}{7} \frac{2}{6} \frac{1}{5}. \end{aligned}$$