Eigth Homework Set — Solutions
Chapter 4

Problem 72 Let A be the stronger team. $P(A \text{ wins in } i \text{ games}) = \binom{i-1}{i-4} 0.6^i 0.4^{i-4}$, for $i = 4, \ldots, 7$. Hence

$$P(A \text{ wins best-of-seven series}) = \sum_{i=4}^{7} \binom{i-1}{i-4} 0.6^4 0.4^{i-4} = 0.7102.$$

Similarly,

$$P(A \text{ wins best-of-three series}) = \sum_{i=2}^{3} \binom{i-1}{i-2} 0.6^4 0.4^{i-2} = 0.6480.$$

Problem 73 Let X be the number of games played in a match. Then $P\{X = i\} = 2\binom{i-1}{i-4}\left(\frac{1}{2}\right)^i$ for $i = 4, \ldots, 7$. Hence, $E[X] = 2\sum_{i=4}^{7} i\binom{i-1}{i-4}\left(\frac{1}{2}\right)^i = 5.8125$.

Problem 77 Let E be the event that right-hand box is emptied while the left-hand box still contains k matches. Then, using a negative binomial random variable with $p = \frac{1}{2}$, $r = N$, and $n = 2N - k$, we see that $P(E) = \binom{2N-k-1}{N-1}\left(\frac{1}{2}\right)^{2N-k}$. Now the desired probability is $2P(E)$.

Problem 78 Let E be the event that a single drawing results in two white and two black balls. Then $P(E) = \frac{\binom{4}{2}\binom{4}{2}}{\binom{8}{4}} = \frac{18}{35}$.

Let X be the number of selections until E occurs. Then

$$P\{X = n\} = \frac{17^{n-1} \cdot 18}{35^n}.$$

Problem 79 (a) $P\{X = 0\} = \frac{\binom{94}{10}}{\binom{100}{10}} = 0.5223$

(b)

$$\begin{aligned} P\{X > 2\} &= 1 - P\{X = 0\} - P\{X = 1\} - P\{X = 2\} \\ &= \frac{\binom{100}{10} - \binom{94}{10} - \binom{6}{1}\binom{94}{9} - \binom{6}{2}\binom{94}{8}}{\binom{100}{10}} = 0.0126 \end{aligned}$$

Chapter 5

Problem 1 (a) We have $1 = \int_{-1}^{1} c(1-x^2)dx = cx\left(1-\frac{x^2}{3}\right)|_{-1}^{1} = \frac{4}{3}c$, so that $c = \frac{3}{4}$.

(b) We have $\int_{-1}^{x} f(y)dy = \frac{3}{4}y\left(1-\frac{y^2}{3}\right)|_{-1}^{x} = \frac{1}{2}+\frac{3}{4}x\left(1-\frac{x^2}{3}\right)$ if $-1 \leq x \leq 1$. Hence,

$$F(x) = \begin{cases} 0 & x < -1, \\ \frac{1}{2}+\frac{3}{4}x\left(1-\frac{x^2}{3}\right) & -1 \leq x \leq 1, \\ 1 & x > 1. \end{cases}$$

Problem 2 Determine C: $\int_0^\infty xe^{-\frac{x}{2}}dx = -2xe^{-\frac{x}{2}}|_0^\infty + \int_0^\infty 2e^{-\frac{x}{2}}dx = (-2x-4)e^{-\frac{x}{2}}|_0^\infty = 4$, so that $C = \frac{1}{4}$.

Now, we have $P\{X \geq 5\} = \int_5^\infty \frac{1}{4}xe^{-\frac{x}{2}} = -(\frac{x}{2}+1)e^{-\frac{x}{2}}|_5^\infty = \frac{7}{2}e^{-\frac{5}{2}}$

Problem 4 (a) $P\{X > 20\} = \int_{20}^\infty \frac{10}{x^2}dx = -\frac{10}{x}|_{20}^\infty = \frac{1}{2}$.

(b)

$$F(x) = \begin{cases} 0 & x < 10 \\ 1-\frac{10}{x} & x \geq 10 \end{cases}$$

(c) Let's assume that lifetimes of the six devices are independent of each other. Let $p = 1 - F(15)$. Then the desired probability is

$$\sum_{i=3}^{6} \binom{6}{i} p^i (1-p)^{6-i}.$$

Problem 5 We want to find C such that $F(C) \geq 0.99$. We have $F(C) = \int_0^C 5(1-x)^4 dx = -(1-x)^5|_0^C = 1-(1-C)^5$. We want $1-(1-C)^5 \geq 0.99$, i.e., $(1-C)^5 \leq 0.01$, hence $C \geq 1-(0.01)^{0.2}$.

Problem 6 (a)

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} xf(x)dx = \frac{1}{4}\int_0^\infty x^2 e^{-\frac{x}{2}}dx \\ &= \frac{1}{4}\left(-2x^2-8x-16\right)e^{-\frac{x}{2}}|_0^\infty = 4 \end{aligned}$$

(b) $E[X] = \int_{-1}^{1} c(1-x^2)x dx = 0$ by symmetry

(c) $E[X] = \int_5^\infty x \frac{5}{x^2} dx = \int_5^\infty \frac{5}{x} = \infty$

Problem 10 (a) Let X be uniform on $[0, 60]$. Then

$$\begin{aligned} & P(\text{passenger goes to } A) \\ &= P\{5 \leq X < 15\} + P\{20 \leq X < 30\} P\{35 \leq X < 45\} \\ &\quad + P\{50 \leq X < 60\} \\ &= \frac{2}{3}. \end{aligned}$$

(b) Same as above.

Problem 12 If service stations are located in A, B, and the center, then the distance between two service stations is 50 miles, so that the expected distance from a service station at the time of a breakdown is

$$\frac{1}{50}\left(\int_0^{25} x dx + \int_{25}^{50} (50-x) dx\right) = \frac{1}{50}\left(\frac{25^2}{2} + 25 \cdot 50 - \frac{50^2}{2} + \frac{25^2}{2}\right) = 12.5.$$

If the service stations are located at mile 25, 50, and 75, then the expected distance from a station at the time of a breakdown is

$$\begin{aligned} & \frac{1}{50}\left(\int_0^{25} x dx + \int_{25}^{37.5} (x-25) x dx + \int_{37.5}^{50} (50-x) dx\right) \\ &= \frac{1}{50}\left(\frac{25^2}{2} + 2\frac{12.5^2}{2}\right) = 9.375. \end{aligned}$$

The second strategy is more efficient.

Problem 13 (a) $P\{X > 10\} = \frac{2}{3}$

(b) $P\{X > 25 | X > 15\} = \frac{P\{X>25\}}{P\{X>15\}} = \frac{\frac{5}{30}}{\frac{15}{30}} = \frac{1}{3}$.