Ninth Homework Set — Solutions
Chapter 5

Problem 15 (a) $P\{X > 5\} = P\left\{\frac{X-10}{6} > \frac{5-10}{6}\right\} = 1 - \Phi\left(-\frac{5}{6}\right) = \Phi\left(\frac{5}{6}\right) = 0.7977$

(b) $P\{4 < X < 16\} = P\left\{-1 < \frac{X-10}{6} < 1\right\} = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6827$

(c) $P\{X < 8\} = P\left\{\frac{X-10}{6} < -\frac{1}{3}\right\} = \Phi\left(-\frac{1}{3}\right) = 1 - \Phi\left(\frac{1}{3}\right) = 0.3695$

(d) $P\{X < 20\} = P\left\{\frac{X-10}{6} < -\frac{10}{6}\right\} = \Phi\left(\frac{5}{3}\right) = 0.9522$

(e) $P\{X > 16\} = P\left\{\frac{X-10}{6} > 1\right\} = 1 - \Phi(1) = 0.1587$

Problem 18 We have $P\{X > 9\} = P\left\{\frac{X-5}{\sigma} > \frac{4}{\sigma}\right\} = 1 - \Phi\left(\frac{4}{\sigma}\right) = 0.2$, so that $\Phi\left(\frac{4}{\sigma}\right) = 0.8$, hence $\frac{4}{\sigma} = 0.85$. This implies that $\sigma = 4.7059$, so that the variance is $\sigma^2 = 22.145$.

Problem 21 Let X be a normal random variable with $\mu = 71$ and $\sigma^2 = 6.25$. Then $P\{X > 74\} = P\left\{\frac{X-71}{2.5} > \frac{3}{2.5}\right\} = 1 - \Phi\left(\frac{6}{5}\right) = 0.1151$.

Moreover, $P\{X > 77 | X \geq 72\} = \frac{P\left\{\frac{X-71}{2.5} > \frac{6}{2.5}\right\}}{P\left\{\frac{X-71}{2.5} \geq \frac{1}{2.5}\right\}} = \frac{1-\Phi\left(\frac{12}{5}\right)}{1-\Phi\left(\frac{2}{5}\right)} = 0.024$.

Problem 22 Let X be normal with $\mu = 0.9$ and $\sigma = 0.003$.

(a) $P\{|X - 0.9| > 0.005\} = P\left\{\frac{|X-0.9|}{0.003} > \frac{5}{3}\right\} = 2 - 2\Phi\left(\frac{5}{3}\right) = 0.095$.

(b) We want $P\left\{\frac{|X-0.9|}{\sigma} > 0.005\right\} = 2 - 2\Phi\left(\frac{0.005}{\sigma}\right) \leq 0.01$, hence $\Phi\left(\frac{0.005}{\sigma}\right) \geq 0.995$, so that $\frac{0.005}{\sigma} \geq 2.58$, hence $\sigma = 0.0019$.

Problem 23 Let X be the number of times the number six appears.

$$P\{149.5 < X < 200.5\} = P\left\{\frac{149.5 - \frac{1000}{6}}{\sqrt{\frac{5000}{36}}} < \frac{X - \frac{1000}{6}}{\sqrt{\frac{5000}{36}}} < \frac{200.5 - \frac{5000}{36}}{\sqrt{\frac{5000}{36}}}\right\}$$
$$= \Phi(2.87) + \Phi(1.46) - 1 = 0.9258.$$

$$P\{X < 149.5\} = P\left\{\frac{X - \frac{800}{5}}{\sqrt{\frac{3200}{25}}} < \frac{149.5 - \frac{800}{5}}{\sqrt{320025}}\right\} = 1 - \Phi(0.92) = 0.1762.$$

Problem 25 Let X be a binomial random variable with $p = 0.05$ and $n = 150$. Then $P\{X \leq 10\} = P\{X \leq 10.5\} = P\left\{\frac{X-7.5}{\sqrt{7.125}} \leq \frac{10.5-7.5}{\sqrt{7.125}}\right\} = \Phi(1.1239) = 0.8695$, using DeMoivre-Laplace.

Problem 28 Let X be the number of lefthanders. Then X is binomial with $p = 0.12$ and $n = 200$. Then

$$\begin{aligned} P\{X \geq 20\} &= P\{X > 19.5\} \\ &= P\left\{\frac{X - 24}{\sqrt{200 \cdot 0.12 \cdot 0.88}} > \frac{19.5 - 24}{\sqrt{200 \cdot 0.12 \cdot 0.88}}\right\} \\ &= 1 - \Phi(-0.9792) = \Phi(0.9792) = 0.8363. \end{aligned}$$

Problem 32 Let X be exponential with parameter $\lambda = \frac{1}{2}$.

(a) $P\{X > 2\} = 1 - F(2) = e^{-1}$

(b) $P\{X > 10 | X > 9\} = P\{X > 1\} = 1 - F(1) = e^{-\frac{1}{2}}$ because X is memoryless.

Problem 33 Let X be an exponential random variable with parameter $\lambda = \frac{1}{8}$. Since X is memoryless, we have $P\{X > t + 8 | X > t\} = P\{X > 8\} = e^{-1}$.

Problem 34 Let X be an exponential random variable with parameter $\lambda = \frac{1}{20}$. Since X is memoryless, we have $P\{X > 30 | X > 10\} = P\{X > 20\} = e^{-1}$.

Let Y be a uniform random variable on $[0, 40]$. Then $P\{X > 30 | X > 10\} = \frac{P\{X>30\}}{P\{X>10\}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$.

Problem 37 Let X be uniformly distributed over $(-1, 1)$.

(a) $P\left\{|X| > \frac{1}{2}\right\} = P\left\{X > \frac{1}{2}\right\} + P\left\{X < -\frac{1}{2}\right\} = \frac{1}{2}$

(b) Let $Y = |X|$. If $y \in (0, 1)$, then $F_Y(y) = P\{Y \leq y\} = P\{-y \leq Y \leq y\} = y$, so that

$$f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 39 Let X be exponential with $\lambda = 1$, and let $Y = \log X$. Then $F_Y(y) = P\{Y \leq y\} = P\{\log X \leq y\} = P\{X \leq e^y\} = 1 - e^{-e^y}$, so that

$$f_Y(y) = e^{y - e^y}.$$

Problem 40 Let X be uniform on $(0, 1)$, and $Y = e^X$. Then, for $1 < y < e$, $F_Y(y) = P\{Y \leq y\} = P\left\{e^X \leq y\right\} = P\{X \leq \log Y\} = \log Y$, so that

$$f_Y(y) = \begin{cases} \frac{1}{y} & 1 < y < e \\ 0 & \text{otherwise} \end{cases}$$

Problem 41 For any $r \in (-A, A)$, we have $F_R r = P\{R \leq r\} = P\{A \sin\theta \leq r\} = P\left\{\theta \leq \arcsin\frac{r}{A}\right\} = \frac{1}{\pi}\arcsin\frac{r}{A}$, so that

$$f_R(r) = \begin{cases} \frac{1}{\pi\sqrt{A^2 - r^2}} & r \in (-A, A) \\ 0 & \text{otherwise} \end{cases}$$