Solutions to Math 461 Test 2, Spring 2007

1. (16 points) A certain basketball player knows that on average he will make 90 percent of his free throw attempts. Use normal approximation to find the probability that in 100 attempts he will be successful at least 93 times.

Solution. Let S_{100} be the number of successful free throws. Then

$$\begin{aligned} P(S_{100} \geq 93) &= P(S_{100} \geq 92.5) = P(\frac{S_{100}-90}{\sqrt{100 \cdot 0.09}} \geq \frac{92.5-90}{\sqrt{100 \cdot 0.09}}) \\ &= P(\frac{S_{100}-90}{3} \geq \frac{2.5}{3}) \approx 1 - \Phi(\frac{5}{6}) = .2033. \end{aligned}$$

2. (14 points) Suppose that X is a Poisson random variable with parameter $\lambda_1 = 1$, that Y is a Poisson random variable with parameter $\lambda_2 = 2$, and that X and Y are independent. Find $P(X = 10|X + Y = 30)$.

Suppose that X is uniformly distributed over the interval $(-2, 2)$. Find the density of the random variable $Y = X^6$.

Solution.

$$\begin{aligned} P(X = 10|X+Y=30) &= \frac{P(X=10, X+Y=30)}{P(X+Y=30)} = \frac{P(X=10, Y=20)}{P(X+Y=30)} \\ &= \frac{P(X=10)P(Y=20)}{P(X+Y=30)} = \frac{e^{-1}\frac{1}{10!}e^{-2}\frac{2^{20}}{20!}}{e^{-3}\frac{3^{30}}{30!}} \\ &= \binom{30}{10}(\frac{1}{3})^{10}(\frac{2}{3})^{20}. \end{aligned}$$

3. (15 points) Suppose that X is uniformly distributed over the interval $(-2, 2)$. Find the density of the random variable $Y = X^6$.

Solution. For $y \in (0, 2^6)$,

$$P(Y \leq y) = P(X^6 \leq y) = P(y^{1/6} \leq X \leq y^{1/6}) = \frac{1}{2}y^{1/6},$$

so the density of Y is

$$f_Y(y) = \begin{cases} \frac{1}{12}y^{-5/6}, & y \in (0, 2^6) \\ 0, & \text{otherwise.} \end{cases}$$

4. (15 points) Suppose that X and Y are independent uniform random variables on $(0, 1)$. Find (a) $P(Y \geq 2X)$; (b) $P(Y \geq 2X|Y \leq 1/2)$.

Solution. By drawing a picture and using geometric considerations we can get
(a) $P(Y \geq 2X) = \frac{1}{4}$.

(b) $P(Y \geq 2X, Y \leq 1/2) = \frac{1}{16}$, $P(Y \leq 1/2) = \frac{1}{2}$. Thus

$$P(Y \geq 2X | Y \leq 1/2) = \frac{P(Y \geq 2X, Y \leq 1/2)}{P(Y \leq 1/2)} = \frac{1}{8}.$$

5. (10 points) The joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{2}x + \frac{1}{4}y, & 0 < x < 1, 0 < y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the marginal density of Y. (b) Find EY and $\text{Var}(Y)$.

Solution. (a) Using the formula $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$, we can easily get

$$f_Y(y) = \begin{cases} \frac{1}{4}(1+y), & 0 < y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $EY = \int_0^2 y\frac{1}{4}(1+y)dy = \frac{7}{6}$. $E(Y^2) = \int_0^2 y^2\frac{1}{4}(1+y)dy = \frac{5}{3}$. Thus $\text{Var}(Y) = \frac{5}{3} - (\frac{7}{6})^2$.

6. (15 points) Suppose that X and Y are independent exponential random variables with parameter $\lambda = 1$. Find the density function of $Z = X/Y$.

Solution. Obviously Z is a positive random variable. For any $z > 0$, we have

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P(X \leq zY) \\ &= \int_0^\infty \int_0^{zy} e^{-x-y} dx dy = 1 - \frac{1}{z+1}. \end{aligned}$$

Taking derivatives we get

$$f_Z(z) = \begin{cases} \frac{1}{(z+1)^2}, & z > 0, \\ 0, & z \leq 0. \end{cases}$$

7. (15 points) The gross daily sales at a certain convenient store is a normal random variable with mean \$1000 and standard deviation \$100. Assume that sales are independent from day to day. Find the probability that the total gross sales in the next 4 days exceeds \$4400.

By the assumption of the problem we know that total gross sales in the next 4 days is a normal random variable with mean 4000 and variance $4 \cdot (100)^2$. So

$$P(X \geq 4400) = P(\frac{X - 4000}{200} \geq \frac{400}{200}) = 1 - \Phi(2) = .0228.$$