Solutions to Math 461 Test 1, Spring 2007

1. (16 points) An electric circuit is as below. The probability that the i-th switch is on is equal to $\frac{1}{i+1}$, $i = 1, 2, 3, 4$. Assume that all switches function independently. (a) Find the probability that electric current can flow from A to B. (b) given that electric current can flow from A to B, find the probability that the third switch is on.

Solution. Let E be the event that electric current can flow from A to B, and let A_i, $i = 1, 2, 3, 4$ be the event that the i-th switch is on. Then $E = ((A_1 \cap A_2) \cup A_3) \cap A_4$. Thus (a)

$$\begin{aligned} P(E) &= P((A_1 \cap A_2) \cup A_3)P(A_4) \\ &= (P(A_1 \cap A_2) + P(A_3) - P(A_1 \cap A_2 \cap A_3)) \cdot \frac{1}{5} \\ &= (\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{3}\frac{1}{4}) \cdot \frac{1}{5} = \frac{3}{40}. \end{aligned}$$

(b)

$$P(A_3|E) = \frac{P(A_3 \cap E)}{P(E)} = \frac{P(A_3 \cap A_4)}{P(E)} = \frac{2}{3}.$$

2. (14 points) Suppose that box I contains 1 black and 1 white marble and that box II contains 2 black and 1 white marble. A box is selected at random, and then a marble is drawn at random from the selected box. (a) Find the probability that the marble is black. (b) find the probability that box I was the selected box given that the marble was black.

Solution. Let E_1 be the event that box I is selected, E_2 the event that box II is selected, and B the event that the marble is black. Then

(a) $P(B) = P(E_1 \cap B) + P(E_2 \cap B) = P(E_1)P(B|E_1) + P(E_2)P(B|E_2) = \frac{1}{2}(\frac{1}{2} + \frac{2}{3}) = \frac{7}{12}$.

(b) $P(E_1|B) = P(E_1 \cap B)/P(B) = P(E_1)P(B|E_1)/(7/12) = \frac{3}{7}$.

3. (15 points) A 7 card poker-hand is random drawn from an ordinary deck of 52 cards. Find the probability that the hand contains at least one card from each of the 4 suits.

Solution. Let E_1 be the event that clubs are missing from the hand, E_2 the event that diamonds are missing, E_3 the event that event that hearts are missing, E_4 the event that spades are missing, and A the event that the hand contains at least one card from each of the 4 suits. Then $A^c = \cup_{i=1}^4 E_i$. Since

$$\begin{aligned} P(\cup_{i=1}^4 E_i) &= \sum_{i=1}^4 P(E_i) - \sum_{i_1<i_2} P(E_{i_1} \cap E_{i_2}) - \sum_{i_1<i_2<i_3} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) \\ &= 4 \cdot \frac{\binom{39}{7}}{\binom{52}{7}} - \binom{4}{2}\frac{\binom{26}{7}}{\binom{52}{7}} + \binom{4}{3}\frac{\binom{13}{7}}{\binom{52}{7}}. \end{aligned}$$

Thus

$$P(A) = 1 - \left(4 \cdot \frac{\binom{39}{7}}{\binom{52}{7}} - \binom{4}{2}\frac{\binom{26}{7}}{\binom{52}{7}} + \binom{4}{3}\frac{\binom{13}{7}}{\binom{52}{7}}\right).$$

4. (15 points) A and B play a series of games. Each game is independently won by A with probability $\frac{2}{3}$ and by B with probability $\frac{1}{3}$. They stop when the total number of wins of one of the players is two greater than that of the other player. Find the probability that a total of 6 games were played.

Solution. In order that a total of 6 games be played, the two players have to win one each of the first two games, and win one each of the next two games, and games 5 and 6 have to be won by one of the players. So the answer is

$$\left(2\cdot\frac{2}{3}\cdot\frac{1}{3}\right)^2\left(\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2\right)=\left(\frac{4}{9}\right)^2\cdot\frac{5}{9}.$$

5. (10 points) Let X be a random variable whose distribution function F is given by

$$F(x)=\begin{cases}0, & x<0,\\ x/3, & 0\le x<1,\\ x/2 & 1\le x<2,\\ 1, & 2\le x.\end{cases}$$

Find (a) $P(\frac{1}{2}\le X\le\frac{3}{2})$; (b) $P(\frac{1}{2}\le X\le 1)$; (c) $P(1\le X<2)$; (d) $P(X>3/2)$; (e) $P(X=1)$.

Solution. (a) $P(\frac{1}{2}\le X\le\frac{3}{2})=F(\frac{2}{3})-F(\frac{1}{2}-)=\frac{3}{4}-\frac{1}{6}$.

(b) $P(\frac{1}{2}\le X\le 1)=F(1)-F(\frac{1}{2}-)=\frac{1}{2}-\frac{1}{6}$.

(c) $P(1\le X<2)=F(2-)-F(1-)=1-\frac{1}{3}$.

(d) $P(X>3/2)=1-F(\frac{2}{3})=1-\frac{3}{4}$.

(e) $P(X=1)=F(1)-F(1-)=\frac{1}{2}-\frac{1}{3}$.

6. (15 points) From a group of 7 freshman, 6 sophomores, 5 juniors and 5 seniors a committee of size 4 is randomly selected. (a) Find the probability that the committee consists of 1 from each class. (b) Find the probability that the committee consists of people all from the same class.

Solution. (a)

$$\frac{\binom{7}{1}\binom{6}{1}\binom{5}{1}\binom{5}{1}}{\binom{23}{4}}.$$

(b)

$$\frac{\binom{7}{4}+\binom{6}{4}+\binom{5}{4}+\binom{5}{4}}{\binom{23}{4}}.$$

7. (15 points) A fair coin is rolled 4 times. Let X be the number of heads and let $Y=\cos(\frac{\pi X}{2})$. (a) Find $P(X\ge 2)$. (b) Find the expectation and variance of Y.

Solution. (a) $P(X\ge 2)=1-P(X=0)-P(X=1)=1-\left(\frac{1}{2}\right)^4-4\cdot\left(\frac{1}{2}\right)^4=\frac{11}{16}$.

(b)

$$\begin{aligned}
EY &= \sum_{i=1}^{4} \cos(\frac{i\pi}{2}) P(X=i) = P(X=0) - P(X=2) + P(X=4) \\
&= \left(\frac{1}{2}\right)^4 - \binom{4}{2}\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 = -\frac{1}{4}.
\end{aligned}$$

$$\begin{aligned}
E[Y^2] &= \sum_{i=1}^{4} \cos^2(\frac{i\pi}{2}) P(X=i) = P(X=0) - P(X=2) + P(X=4) \\
&= \left(\frac{1}{2}\right)^4 + \binom{4}{2}\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 = \frac{1}{2}.
\end{aligned}$$

Thus $\mathrm{Var}(Y) = \frac{1}{2} - (-\frac{1}{4})^2$.