Twelfth Homework Set — Solutions

Chapter 6

Problem 49 Let $X_1, \ldots, X_5$ be independent exponential random variables with parameter λ.

(a)

$$
\begin{aligned}
P\{\min(X_1, \ldots, X_5) \leq a\} &= 1 - P\{\min(X_1, \ldots, X_5) > a\} \\
&= 1 - P\{X_1 > a, \ldots, X_5 > a\} \\
&= 1 - P\{X_1 > a\} \cdots P\{X_5 > a\} \\
&= \begin{cases} 1 - (e^{-\lambda a})^5 & a > 0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

(b)

$$
\begin{aligned}
P\{\max(X_1, \ldots, X_5) \leq a\} &= P\{X_1 \leq a, \ldots, X_5 \leq a\} \\
&= P\{X_1 \leq a\} \cdots P\{X_5 \leq a\} \\
&= \begin{cases} (1 - e^{-\lambda a})^5 & a > 0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Chapter 7

Problem 5 If (X, Y) is the location of the accident, then X and Y are uniform random variables on $(-\frac{3}{2}, \frac{3}{2})$. Let $D = |X| + |Y|$. Then

$$
\begin{aligned}
E[D] &= E[|X|] + E[|Y|] = 2E[|X|] \\
&= 2\int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{|x|}{3} dx = \frac{4}{3}\int_0^{\frac{3}{2}} x dx \\
&= \frac{4}{3} \cdot \frac{9}{8} = \frac{3}{2}.
\end{aligned}
$$

Problem 6 Let X_i be the outcome of the i-th roll of the die, for $i = 1, \ldots, 10$, and note that $E[X_i] = \frac{7}{2}$. Let $X = X_1 + \cdots + X_{10}$. Now $E[X] = E[X_1] + \cdots + E[X_{10}] = 10E[X_1] = 35$.

Problem 7 (a) Let X_i be one if both A and B choose the i-th object, for $i = 1, \ldots, 10$. Then $E[X_i] = P\{X_i = 1\} = \left(\frac{3}{10}\right)^2 = \frac{9}{100}$. Now, the expected number of objects chosen by both A and B is $E[X_1] + \cdots + E[X_{10}] = 10E[X_1] = 0.9$.

(b) Let Y_i be one if neither A nor B choose the i-th object. Then $E[Y_i] = P\{Y_i = 1\} = \left(\frac{7}{10}\right)^2 = \frac{49}{100}$, so that $E[Y_1 + \cdots + Y_{10}] = 10E[Y_1] = 4.9$.

(c) Let Z_i be one if either A or B (but not both) chooses the i-th object. Then $E[Z_i] = P\{Z_i = 1\} = 2\frac{3}{10}\frac{7}{10} = \frac{21}{50}$. Now, $E[Z_1 + \cdots + Z_{10}] = 10E[Z_1] = \frac{21}{5} = 4.2$.

Problem 8 Following the hint, let X_1 be one if the i-th arrival sits at a previously unoccupied table. Then $E[X_i] = P\{X_i = 1\} = (1-p)^{i-1}$, so that

$$E[X_1 + \cdots + X_N] = \sum_{i=1}^{N}(1-p)^{i-1} = \frac{1-(1-p)^N}{1-(1-p)} = \frac{1-(1-p)^N}{p}.$$

Problem 11 Let X_i be one if the i-th outcome differs from the $(i-1)$-th outcome, for $i = 2, \ldots, n$. We have $E[X_i] = P\{X_i = 1\} = 2p(1-p)$, so that $E[X_2 + \cdots + X_n] = 2(n-1)p(1-p)$.

Problem 18 Let X_i be one if the i-th card is a match, for $i = 1, \ldots, 13$, and let $X = X_1 + \cdots + X_{52}$. Then $P\{X_i = 1\} = \frac{1}{13}$, so that $E[X] = 52E[X_1] = \frac{52}{13} = 4$.

Problem 19 (a) If X is the number of insects caught before a type 1 catch, then $(X+1)$ is geometric with parameter P_1, so that $E[X] = \frac{1}{P_1} - 1$.

(b) Let Y_i be one if an insect of type i is caught before an insect of type 1, for $i = 2, \ldots, r$. Then $Y = Y_2 + \cdots + Y_r$ is the number of insects caught before an insect of type 1. We have $E[Y_i] = P\{Y_i = 1\} = \frac{P_i}{P_i + P_1}$, so that

$$E[Y] = \sum_{i=2}^{r} \frac{P_i}{P_i + P_1}.$$

Problem 21 (a) Let X be the number of days of the year that are birthdays of exactly 3 people. For $i = 1, \ldots, 365$, let $X_i = 1$ if the i-day is the birthday of eaxctly 3 people and $X_i = 0$ otherwise. Then $X = \sum_{i=1}^{365} X_i$. Since for each i,

$$EX_i = P(X_i = 1) = \binom{100}{3} \left(\frac{1}{365}\right)^3 \left(\frac{364}{365}\right)^{97},$$

we get that

$$EX = 365 \binom{100}{3} \left(\frac{1}{365}\right)^3 \left(\frac{364}{365}\right)^{97}.$$

(b) Let Y be the number of distinct birthdays. For $i = 1, \ldots, 365$, let $Y_i = 1$ if the i-day is someone's birthday and $Y_i = 0$ otherwise. Then $Y = \sum_{i=1}^{365} Y_i$. Since for each i,

$$EY_i = P(Y_i = 1) = 1 - P(Y_i = 0) = 1 - \left(\frac{364}{365}\right)^{100},$$

we get that

$$EY = 365 \left[1 - \left(\frac{364}{365}\right)^{100}\right].$$

Problem 30 Note that $E[X^2] = E[Y^2] = \mathrm{Var}(X) + E[X]^2 = \sigma^2 + \mu^2$. Now we conclude that

$$E\left[(X - Y)^2\right] = E\left[X^2\right] - 2E[X]E[Y] + E\left[Y^2\right] = 2\sigma^2,$$

using the fact that X and Y are independent.

Problem 31 Let X_i be the outcome of the i-th roll of the die, for $i = 1, \ldots, 10$. Then $\mathrm{Var}(X_i) = \frac{35}{12}$, so that

$$\mathrm{Var}(X_1 + \cdots + X_{10}) = 10 \cdot \frac{35}{12} = \frac{175}{6}.$$

Problem 33 (a)

$$E\left[(2 + X)^2\right] = 4 + 4E[X] + E\left[X^2\right] = 8 + \mathrm{Var}(X) + E[X]^2 = 14.$$

(b)

$$\mathrm{Var}(4 + 3X) = 9\mathrm{Var}(X) = 45.$$

Problem 38 We have

$$\begin{aligned}
E[XY] &= \int_0^\infty \int_0^x 2ye^{-2x} dy dx = \int_0^\infty x^2 e^{-2x} dx = \frac{1}{4}, \\
E[X] &= \int_0^\infty \int_0^x 2e^{-2x} dy dx = \int_0^\infty 2xe^{-2x} dx = \frac{1}{2}, \quad \text{and} \\
E[Y] &= \int_0^\infty \int_0^x \frac{2y}{x} e^{-2x} dy dx = \int_0^\infty xe^{-2x} dx = \frac{1}{4}.
\end{aligned}$$

Hence,

$$\text{Cov}(X, Y) = \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$